



Institute for Empirical Research in Economics
University of Zurich

Working Paper Series
ISSN 1424-0459

Working Paper No. 101

**On the Efficiency of Monetary Exchange:
How Divisibility of Money Matters**

Aleksander Berentsen and Guillaume Rocheteau

January 2002

On the Efficiency of Monetary Exchange: How Divisibility of Money Matters.*

Aleksander Berentsen[†]

Economics Department, University of Basel, Switzerland

Guillaume Rocheteau[‡]

School of Economics, Australian National University, Australia.

First version: July 25, 2000

Revised version: August 28, 2001

Abstract

We use alternative assumptions about the divisibility of goods and money and the ability of agents to use lotteries on money to investigate to what extent the indivisibility of money is the cause for the typically inefficient production and consumption decisions in search-theoretic models of money. Our framework potentially generates three types of inefficiencies: the *no-trade* inefficiency, where no trade takes place even though it would be socially efficient to trade; and the *too-much-trade* and *too-little-trade* inefficiencies, where the quantities produced and exchanged are either larger or smaller than what the solution to a social planner's problem would mandate. It is shown that while the no-trade and the too-much-trade inefficiencies are caused by the indivisibility of money, the too-little-trade inefficiency remains even when money is divisible unless it is sufficiently valued.

JEL: E00, D83, E52

Keywords: Money, Indivisibility, Search.

*This paper is forthcoming in the Journal of Monetary Economics. It has benefited from discussions with Randall Wright. We also thank Stefan Arping, Ernst Baltensperger, Peter Rupert, Simon Lörtscher, an anonymous referee, the participants in the money workshop in Paris (May 2000), and the attendees of the Conference on Monetary Economics at the Federal Reserve Bank of Cleveland (August 2001). The first author gratefully acknowledges financial support from a grant received by the Swiss National Science Foundation.

[†]Address: University of Basel, Economics Department (WWZ), Petersgraben 51, 4003 Basel, Switzerland. E-mail: aleksander.berentsen@unibas.ch

[‡]Address: Copland building, School of Economics, Australian National University, Canberra, ACT 0200, Australia. E-mail: guillaume.rocheteau@anu.edu.au

1 Introduction

A well-known feature of search-theoretic models of money is that the outcome is generically inefficient in the sense that the quantities exchanged differ from the solution to a social planner's problem. Some models predict that prices are too high. Models with this property include the divisible money and divisible goods model of Shi (1997, 1999). Other models find that prices can be too low. For certain parameter values, the divisible goods and indivisible money models of Trejos and Wright (1995) and Shi (1995) display this outcome. Moreover, in the indivisible money and indivisible goods model of Kiyotaki and Wright (1991), in some meetings there is no production and consumption even though it would be socially efficient to trade.¹

What lies behind these inefficiencies? Although each paper in question explains the reason for the observed inefficiencies, there is no obvious basis for a comparison of their results. The explanations for these inefficiencies range from the decentralized nature of the price formation (Trejos and Wright, 1995; Shi, 1995) to the time-consuming exchange process (Kiyotaki and Wright, 1991; Shi, 1997) or the details of the bargaining protocol.² We are therefore left with apparently unrelated explanations, which limits our understanding of the search-theoretic approach.

This paper explores the nature of these inefficiencies by providing a common framework for assessing the different models. We assume that the economy is composed of many commodities, and that for each commodity there is a continuum of differentiated varieties. Agents have idiosyncratic tastes for these varieties, and they choose strategies for determining when (and sometimes how much) to produce, to trade, and to consume. Because of the randomness of the matching process, in a single-coincidence meeting the buyer's preference for the seller's variety is represented by a random variable ε : this is what we call a stochastic mismatch problem.

Basically, the paper's methodology is to introduce this stochastic mismatch problem into the different random-matching models of money in order to identify the consequences of the alternative assumptions about the divisibility of money and goods. First, we compare the indivisible goods and indivisible money model of Kiyotaki and Wright (1991, 1993) with the same model when agents use lotteries on indivisible money to determine the terms of trade.³

¹The no-trade inefficiency is not explicitly mentioned in Kiyotaki and Wright (1991), but it is discussed in Boldrin et al. (1993).

²According to Trejos and Wright (1995, p. 130), the fact that the quantity exchanged is too high is related to the bargaining protocol and to the ability of the traders to meet other traders during the negotiation. Furthermore, even when the traders do not meet other traders while bargaining, if the bargaining power of buyers is sufficiently close to one, the quantity exchanged can be inefficiently large (Wright, 1999).

³In contrast to Berentsen et al. (2000), who consider randomization over both indivisible goods and

Second, we compare the divisible goods and indivisible money model of Trejos and Wright (1995) and Shi (1995), the same model with lotteries on money, and the divisible money and divisible goods model of Shi (1997, 1999). Introducing lotteries on money, like the study of divisible money, will be useful for identifying the origin of the inefficiencies that arise in indivisible money models.

Our framework potentially generates three types of inefficiencies that can arise simultaneously. First, there is what we call a *no-trade* inefficiency: in some meetings there is no production and consumption even though trading would be efficient. Second, there is a *too-much-trade* inefficiency: in some meetings the quantities produced and exchanged are larger than what the solution to a social planner's problem would dictate. Third, there is a *too-little-trade* inefficiency: in some meetings production is too small relative to the planner's solution.

The following results emerge from the analysis. First, the no-trade and the too-much-trade inefficiencies are caused by the indivisibility of money. They are present neither in the divisible money model nor in indivisible money models when lotteries are allowed. Second, the too-little-trade inefficiency appears in all models. This inefficiency is due to the impatience of the traders and the time-consuming nature of the exchange process. Third, the set of meetings where the quantities traded are inefficiently low is smaller, and the purchasing power of money larger, when money is divisible. Fourth, the bargaining procedure, particularly the bargaining power of buyers and sellers, plays a much less important role than we expected. In fact, we find that the results stated above do not depend on the bargaining power of the buyers.

We also show how the divisibility of money affects the Pareto frontier of the bargaining set in a match between a buyer and a seller. The bargaining set of the bargaining game with divisible money (or with indivisible money and lotteries) always contains the bargaining set of the game with indivisible money. The reason for this result is that indivisible money, in contrast to divisible money (or lotteries), transfers utility imperfectly. Intuitively, when money is perfectly divisible, the buyer makes a monetary transfer to compensate the seller for his production cost. Because money is *equally* valued by the buyer and the seller, money is a perfect means to transfer utility. When money is indivisible, however, the seller adjusts the quantity he produces to compensate the buyer for the indivisible unit of money. In contrast to a transfer of divisible money, real production is an inefficient means to transfer utility, because of the decreasing marginal utility of consumption. Graphically, the fact that

indivisible money, we will only consider lotteries on money. This restriction allows us to focus on the implications of indivisible money, because it introduces a notion of divisible money without affecting the indivisibility of goods.

divisible money (or indivisible money with lotteries) transfers utility perfectly is illustrated by a Pareto frontier which is partially linear with divisible money (or with indivisible money and lotteries) and strictly concave with indivisible money without lotteries.

Our framework makes the divisibility of money *visible*. In our divisible money model, we find a threshold $\tilde{\varepsilon}$ such that in a single-coincidence meeting if a buyer's valuation for the seller's good is below $\tilde{\varepsilon}$, he only spends a *fraction* of his money holdings, and if his valuation is above, he spends his entire money holdings. Moreover, if $\varepsilon < \tilde{\varepsilon}$, a socially efficient quantity of the good is produced and exchanged, and if $\varepsilon \geq \tilde{\varepsilon}$, an inefficiently small quantity is traded. In contrast, in Shi's (1997, 1999) divisible money model, buyers in each meeting spend their entire money holdings, and they always receive inefficiently small quantities of goods in exchange.⁴ Interestingly, we find that the model with indivisible money, divisible goods, and lotteries and the divisible money model yield very similar results, because in the lottery model there is also a threshold $\tilde{\varepsilon}$ that has the same properties as above.

We also show that the indivisibility of money can generate a welfare-improving role for inflation. In the indivisible goods and indivisible money model, inflation has a positive effect on welfare, because inflation induces agents to spend their indivisible units of money more quickly. This effect is also present in the divisible goods and indivisible money model. In this model, however, an increase in the growth rate of the money supply has also an intensive effect by reducing the quantities of goods traded in each match. In the divisible money model or the indivisible money model with lotteries, inflation always reduces welfare.

Our paper is related to search models of money that study the role of the divisibility assumption in monetary exchange. Taber and Wallace (1999) consider the divisibility of money in the setups due to Shi (1995) and Trejos and Wright (1995) by relaxing the restrictions on agents' money holdings. Money is said to be twice as divisible if both the upper bound on money holdings and the total number of money units are doubled. The paper is also related to all search models of money that discuss the efficiency of monetary exchange at some point. As far as we know, however, our paper is the first one that makes a comparison across the different types of models. Finally, Jafarey and Masters (2000) study a random-matching model of indivisible money where agents have idiosyncratic preferences for each others' goods. They find similar inefficiencies to ours. However, they concentrate on different issues from the ones we treat.

Section 2 presents the assumptions shared by the different models used in this paper. Section 3 introduces the mismatch problem into the indivisible money models. Section 4 studies the divisible goods and divisible money model, Section 5 shows how the divisibility

⁴Note also, that in contrast to Shi's (1997, 1999) framework, in our model the velocity of money is endogenous and strictly increasing in the rate of inflation.

of money affects the Pareto frontier of the bargaining set, and Section 6 concludes.

2 The environment

The economy consists of a continuum of infinitely lived households of measure one, denoted by \mathcal{C} , that specialize in consumption and production. There are $H \geq 3$ types of goods and H types of households. Households are uniformly distributed among types, so that the measure of households of some given type h is $z \equiv \frac{1}{H}$. For each type of good, there is a continuum of varieties represented by a circle of circumference 2 denoted by \mathcal{C}_h . Household $i \in \mathcal{C}_h$ produces the variety i of type h , and it derives utility from consuming all varieties of type $h + 1 \pmod{H}$. Because each household can be identified by the good it produces, we have $\mathcal{C} = \bigcup_{h \in \{1, \dots, H\}} \mathcal{C}_h$. Denote by Q the set of feasible quantities that a household can produce. We will consider two cases: the indivisible goods model, where $Q = \{0, 1\}$, and the divisible goods model, where $Q = \mathcal{R}^+$.

The mismatch problem is similar to that of Kiyotaki and Wright (1991). The most preferred variety of household $i \in \mathcal{C}_h$ is i^* chosen at random on $\mathcal{C}_{h+1} \pmod{H}$. If we draw at random a variety j from \mathcal{C}_{h+1} , the length l of the arc between i^* and j is uniformly distributed on $[0, 1]$. Accordingly, if household i consumes only varieties within distance \bar{l} of its most preferred variety i^* , the probability that a randomly chosen variety will be consumed by the household is \bar{l} .

The function mapping the distance between the variety that is consumed and the most preferred variety, l , and the quantity consumed, q , into utility is continuous in both arguments, strictly decreasing in l , and increasing in q . We adopt the following function:⁵

$$\mathcal{U}(l, q) = \varepsilon(l) u(q)$$

where ε is strictly decreasing and twice differentiable, and satisfies $\varepsilon(0) = \varepsilon_{\sup}$ and $\varepsilon(1) = 0$. Furthermore, we assume that u is increasing and twice differentiable, and satisfies $u(0) = 0$, $u(1) = U$, $u'' < 0$, and $u'(0) = \infty$. The probability that ε is less than $x \in [0, \varepsilon_{\sup}]$ for a variety chosen at random is equal to

$$\mathbb{P}[\varepsilon(l) \leq x] = \mathbb{P}[l \geq \varepsilon^{-1}(x)] = 1 - \varepsilon^{-1}(x) \equiv F(x)$$

where $F(\cdot)$ is a cumulative distribution with density f . Note that our assumptions on preferences and production rule out barter trades.⁶

⁵Note that the analysis would hold for any utility function satisfying $\frac{\partial \mathcal{U}(l, q)}{\partial q} > 0$, $\frac{\partial \mathcal{U}(l, q)}{\partial l} < 0$, $\frac{\partial^2 \mathcal{U}(l, q)}{\partial q^2} < 0$, and $\frac{\partial^2 \mathcal{U}(l, q)}{\partial q \partial l} < 0$.

⁶In Berentsen and Rocheteau (2000) we study a related environment where in each meeting there is a double coincidence of real wants.

A producer receives disutility $c(q)$ from producing q units of a good. For the divisible goods model, we will assume that $c(q) = q$. For the indivisible goods model, let $c(1) = C$ with $C < \varepsilon_{\sup} U$. Goods cannot be stored, and production is instantaneous. Finally, the discount factor is $\beta = \frac{1}{1+r}$ with $r > 0$.

In addition to the consumption goods described above there is an intrinsically worthless, storable object called fiat money. We will again consider two cases: the *indivisible* money model, where each household consists of one individual, and the *divisible* money model, where each household consists of a continuum of members. For the indivisible money model, a fraction $N \in (0, 1)$ of all agents are initially endowed with one unit of money and each agent has a single unit storage capacity. For the divisible money model, each household is initially endowed with M units of divisible money and the household evenly distributes this amount among a fraction N of its members. In both models, therefore, the fraction of agents in the market endowed with money is exogenous and equal to N , and the fraction of agents without money is $1 - N$. We call agents with money *buyers*, and agents without *sellers*. A buyer attempts to exchange money for consumption goods, and a seller attempts to produce goods for money.

Buyers and sellers meet pairwise and at random. Our assumptions about technology and preferences rule out meetings with double coincidence of real wants. Moreover, buyers only buy varieties that lie in the set $A \subseteq [0, \varepsilon_{\sup}]$, which is determined endogenously. Consequently, for a buyer of type h who is matched to a seller, the probability of a successful trade is $z \int_A dF(x)$, where $z = \frac{1}{H}$ is the probability that the seller is of type $h + 1$, and $\int_A dF(x)$ is the probability that he produces a variety in A .

Throughout the paper we assume that in a match the buyer makes a take-it-or-leave-it offer and the seller accepts the offer if made no worse off by accepting. We adopt this simplifying assumption because in Appendix B we show that our results also hold when the terms of trade are determined through general Nash bargaining.

3 Indivisible money

The aim of this section is to identify the inefficiencies that are present in indivisible money models. For this purpose we introduce the same mismatch problem in the indivisible money and indivisible goods model of Kiyotaki and Wright (1991, 1993) and the indivisible money and divisible goods model of Shi (1995) and Trejos and Wright (1995). We will also study the outcome in these models when agents are allowed to use lotteries on money to determine the terms of trades. Like the study of the divisible money model in Section 4, analyzing lotteries on money will be useful for identifying the origin of the inefficiencies in search models of

money.

3.1 Indivisible goods without lotteries

Time is continuous, agents cannot hold more than one object at a time, and the terms of trades are exogenous: one unit of money buys one indivisible commodity. Only individuals without money (sellers) are willing to produce.⁷ Buyers and sellers meet randomly according to a Poisson process with arrival rate normalized to one. Thus, the probability per unit time that a buyer of type h meets a seller of type $h + 1$ is $z(1 - N)$, and the probability per unit time that a seller of type h meets a buyer of type $h - 1$ is zN .

To introduce a notion of inflation, we assume that a buyer's indivisible unit of money is *confiscated* at rate μ , and that sellers receive one unit of money at rate $\delta = \frac{\mu N}{1-N}$.⁸ Consequently, the quantity of money is constant and equal to N . The value function of buyers satisfies the following Bellman equation:

$$rV_B = z(1 - N) \int_0^{\varepsilon_{\sup}} \max(\varepsilon U + V_S - V_B, 0) dF(\varepsilon) - \mu(V_B - V_S) \quad (1)$$

The flow return to a buyer, rV_B , equals the sum of two terms. The first term is the rate at which the buyer meets appropriate sellers, $z(1 - N)$, times the expected gain of a match, which equals $\int_0^{\varepsilon_{\sup}} \max(\varepsilon U + V_S - V_B, 0) dF(\varepsilon)$. In a single-coincidence meeting, either the buyer spends his unit of money, which yields $\varepsilon U + V_S - V_B$, or he remains a buyer, which yields no surplus. The second term is the rate at which a buyer's money is confiscated, μ , times the confiscation loss, $V_B - V_S$.

Because a buyer's utility is increasing in ε , the set of acceptable varieties is $A = [\bar{\varepsilon}, \varepsilon_{\sup}]$, where $\bar{\varepsilon}$ is a reservation value for the taste index that satisfies

$$\bar{\varepsilon}U = V_B - V_S \quad (2)$$

If $\varepsilon \geq \bar{\varepsilon}$, the buyer is willing to buy the variety. Denote by \bar{E} the reservation value for the average buyer. The probability that a variety of good $h + 1$ is accepted by a buyer of type h chosen at random is $1 - F(\bar{E})$. Accordingly, the value functions of sellers satisfy the following Bellman equation:

$$rV_S = zN [1 - F(\bar{E})] (-C + V_B - V_S) + \delta(V_B - V_S) \quad (3)$$

⁷Buyers never produce, because they cannot hold more than one unit of money, and because there are no barter meetings.

⁸This mechanism was introduced by Li (1995). It is a proxy for inflation, because it reduces the real value of money, and because the probability that the money is confiscated is proportional to the length of time that the buyer holds the money.

The flow return to a seller, rV_S , equals the sum of two terms. The first term is the rate at which the seller of type h meets a buyer of type $h - 1$, zN , times the probability that the buyer accepts the trade, $1 - F(\bar{\varepsilon})$, times the expected gain from trading, producing, and becoming a buyer, $-C + V_B - V_S$. The second term is the rate at which the seller receives a unit of money, δ , times the gain of becoming a buyer, $V_B - V_S$.

Existence of a monetary equilibrium requires that sellers be willing to produce for money:

$$V_B - V_S \geq C. \quad (4)$$

In the following we only consider symmetric Nash equilibria, where $\bar{\varepsilon} = \bar{E}$.

Definition 1 *For the indivisible goods and indivisible money model, a monetary equilibrium is a triplet $(V_B, V_S, \bar{\varepsilon})$ that satisfies equations (1)–(3) and participation constraint (4).*

To compare the outcome of the decentralized economy with the allocation that a planner would choose in order to maximize welfare, consider the following ex ante Pareto criterion: $W = r(NV_B + (1 - N)V_S)$. Welfare is the expected permanent income of a single agent before money is distributed. Equations (1) and (3) yield

$$W = zN(1 - N) \int_{\bar{\varepsilon}}^{\varepsilon_{\sup}} [\varepsilon U - C] dF(\varepsilon)$$

Welfare is maximized at $\bar{\varepsilon}^* = C/U$. From a social point of view buyers should accept any trade with a positive surplus ($\varepsilon U - C \geq 0$), because for the society it does not matter who holds the money.

Proposition 1 *Consider the model with indivisible goods and indivisible money. Then, there exists a critical value ε_0 , defined in the proof, such that the following is true:*

- (i) *If $C/U > \varepsilon_0$, no monetary equilibrium exist.*
- (ii) *If $C/U \leq \varepsilon_0$, there exists a monetary equilibrium.*

Furthermore, $\frac{C}{U} < \varepsilon_0 \Rightarrow \bar{\varepsilon} > C/U$.

Proposition 1 establishes the existence of a monetary equilibrium in the indivisible goods and indivisible money model if the ratio C/U is below the critical value ε_0 . The critical value ε_0 is decreasing in the inflation rate μ , in the rate r of time preference, and in the measure N of buyers.

The key result is that the frequency of trades is generically too low ($\bar{\varepsilon} > C/U$). If $\varepsilon \in [C/U, \bar{\varepsilon})$, agents do not trade even though it would be socially efficient to trade. Buyers are too choosy; they fail to internalize the positive effect of spending money on other market participants. Indeed, a buyer's private gain from a successful trade is $\varepsilon U + V_S - V_B$, whereas

the social gain is $\varepsilon U - C$. The private gain and the social gain coincide if and only if $V_B - V_S = C$. Thus, this condition is satisfied when sellers are just indifferent between accepting and refusing money, which only happens when $\varepsilon_0 = \frac{C}{U}$.

One can remove the no-trade inefficiency by choosing a sufficiently high inflation rate. In Appendix A1 we show that a higher inflation rate reduces the reservation value, because waiting for a better trading opportunity becomes more costly. This is the “hot potato” effect of inflation mentioned by Li (1995). The welfare-maximizing inflation is the value of μ satisfying $\bar{\varepsilon} = C/U$: it is the maximum level that does not destroy the monetary equilibrium.

Finally, as shown in the Appendix, equations (1), (2), and (3) implicitly define a reaction function $\bar{\varepsilon} = \mathcal{R}(\bar{E})$, which is increasing in \bar{E} if $C/U < \varepsilon_0$. This illustrates the presence of strategic complementarities: the best response of a single agent is to increase his reservation value if all other agents increase theirs. Consequently, multiple equilibria may occur.

3.2 Indivisible goods with lotteries on money

We now allow agents to use lotteries on money to determine the terms of trade.⁹ Bargaining over lotteries on money means bargaining over the probability τ_ε that the unit of money changes hands. We again restrict our attention to take-it-or-leave-it offers (for the generalized Nash solution, see Appendix B1). With lotteries on money, the value functions of buyers and sellers satisfy the following generalized versions of (1) and (3):

$$rV_B = z(1 - N) \int_{\bar{\varepsilon}}^{\varepsilon_{\sup}} [\varepsilon U - \tau_\varepsilon (V_B - V_S)] dF(\varepsilon) - \mu (V_B - V_S) \quad (5)$$

$$rV_S = zN \int_{\bar{E}}^{\varepsilon_{\sup}} [-C + \tau_\varepsilon (V_B - V_S)] dF(\varepsilon) + \delta (V_B - V_S) \quad (6)$$

When a buyer meets an appropriate seller, the buyer receives the good and delivers his unit of money with probability τ_ε . Thus, his expected surplus from the trade is $\varepsilon U - \tau_\varepsilon (V_B - V_S)$. Because buyers make take-it-or-leave-it offers, we have $-C + \tau_\varepsilon (V_B - V_S) = 0$ for all ε . Consequently, the probability that money changes hands is constant and equal to

$$\tau_\varepsilon = \tau \equiv \frac{C}{V_B - V_S} \quad \forall \varepsilon \in [\bar{\varepsilon}, \varepsilon_{\sup}] \quad (7)$$

Existence of a monetary equilibrium requires that τ be not larger than one, i.e., that the seller’s participation constraint (4) be satisfied. Because a buyer’s surplus is increasing in ε , buyers trade if and only if $\varepsilon \geq \bar{\varepsilon}$, where $\bar{\varepsilon}$ is a reservation value that satisfies

$$\bar{\varepsilon} U - \tau (V_B - V_S) = 0 \quad (8)$$

⁹In contrast to Berentsen et al. (2000), we only consider lotteries on money, because this allows us to introduce a notion of divisible money without affecting the indivisibility of goods.

Definition 2 *For the indivisible goods and indivisible money model with lotteries on money, a monetary equilibrium is a list $(V_B, V_S, \bar{\varepsilon}, \tau)$ satisfying equations (5)–(8) and $\tau \leq 1$.*

From (5) and (6), welfare is maximized if $\bar{\varepsilon} = \frac{C}{U}$. Note that τ is irrelevant: For the society it does not matter how often money changes hands.

Proposition 2 *Consider the model with indivisible goods, indivisible money, and lotteries. Then there is a critical value ε_0 , defined in the proof, such that the following is true:*

- (i) *If $C/U > \varepsilon_0$, no monetary equilibrium exists.*
- (ii) *If $C/U \leq \varepsilon_0$, a unique monetary equilibrium exists with $\bar{\varepsilon} = \frac{C}{U}$.*

The probability that the unit of money changes hands is

$$\begin{aligned} \tau_\varepsilon &= \tau = \frac{(r + \mu + \delta) C/U}{z(1 - N) \int_{C/U}^{\varepsilon_{\sup}} [1 - F(\varepsilon)] d\varepsilon} & \text{if } \varepsilon \in [C/U, \varepsilon_{\sup}] \\ \tau_\varepsilon &= 0 & \text{if } \varepsilon < C/U \end{aligned}$$

The key result is that the monetary equilibrium is efficient, because whenever there is a positive surplus in a match, the buyer proposes an offer that exploits the entire surplus and this offer is not refused by the seller.¹⁰ Accordingly, the frequency of trades is at its efficient level, i.e., $\bar{\varepsilon} = \frac{C}{U}$. Moreover, lotteries on money remove the strategic complementarities that are present without lotteries, and hence the possibility of multiple equilibria. Also, in contrast to the model without lotteries, inflation has no welfare-improving role, because it cannot increase the frequency of trades. Rather, inflation is detrimental, because it decreases the critical value ε_0 and therefore makes the existence of a monetary equilibrium less likely. Note that the model with lotteries and the same model without lotteries have the same critical value ε_0 .

3.3 Divisible goods without lotteries

In this subsection we explore how the mismatch problem affects the outcome when the terms of trade are endogenized along the lines of Shi (1995) and Trejos and Wright (1995). Suppose, therefore, that goods are divisible but money is still indivisible, and that buyers make a take-it-or-leave-it offer to the seller about the quantity that the seller has to produce for one unit of money. Let q_ε be the amount produced by a seller in exchange for one unit of money when the buyer's valuation for the good is $\varepsilon u(q_\varepsilon)$. Expected lifetime utilities of

¹⁰Appendix B1 shows that efficiency is also attained when the terms of trade are determined through generalized Nash bargaining.

buyers and sellers obey the following Bellman equations in continuous time:

$$rV_B = z(1-N) \int_0^{\varepsilon_{\sup}} \max[\varepsilon u(q_\varepsilon) + V_S - V_B, 0] dF(\varepsilon) + \mu(V_S - V_B) \quad (9)$$

$$rV_S = zN \int_A [-q_\varepsilon + V_B - V_S] dF(\varepsilon) + \delta(V_B - V_S) \quad (10)$$

where $A \subseteq [0, \varepsilon_{\sup}]$ is the set of varieties that money holders are willing to buy. Equations (9) and (10) have similar interpretations to equations (1) and (3). The main difference is that the first term of the right-hand side of (10) is zero, because sellers do not get any benefit from trading with buyers. This is a consequence of the take-it-or-leave-it offers that satisfy

$$q_\varepsilon = q \equiv V_B - V_S \quad \forall \varepsilon \in A \quad (11)$$

Because buyers extract all the surplus from the match, and because the surplus of the seller does not depend on the quality of the match, the terms of trade do not depend on the taste index ε .

The surplus of the buyer, $\varepsilon u(q) + V_S - V_B$, is increasing in ε . Consequently, the reservation property holds, and the set of acceptable varieties is $A = [\bar{\varepsilon}, \varepsilon_{\sup}]$, where the reservation value $\bar{\varepsilon}$ satisfies

$$\bar{\varepsilon} u(q) = q \equiv V_B - V_S \quad (12)$$

Definition 3 *For the divisible goods and indivisible money model, a monetary equilibrium is a list $(V_B, V_S, \bar{\varepsilon}, q)$ that satisfies equations (9)–(12) and $q > 0$.*

Again, we compare the outcome of the decentralized economy with the allocation that a social planner would choose in order to maximize social welfare. Equations (9) and (10) imply that welfare equals

$$W = z(1-N)N \int_{\bar{\varepsilon}}^{\varepsilon_{\sup}} [\varepsilon u(q_\varepsilon) - q_\varepsilon] dF(\varepsilon)$$

By maximizing W with respect to $\bar{\varepsilon}$ and q_ε , we find that the first-best allocation satisfies

$$\begin{aligned} \bar{\varepsilon} &= 0 \\ \varepsilon u'(q_\varepsilon) &= 1 \quad \forall \varepsilon \in [0, \varepsilon_{\sup}] \end{aligned}$$

The first equation states that buyers of type h should consume all varieties of type $h + 1$. The second equation states that the quantity produced, exchanged, and consumed should equalize the marginal utility of the buyer to the marginal cost of the seller. Denote efficient quantities by q_ε^* .

Proposition 3 *Consider the model with divisible goods and indivisible money. Then a unique monetary equilibrium exists with $\bar{\varepsilon} \in (0, \varepsilon_{\text{sup}})$. Furthermore, there is a critical value $\tilde{\varepsilon}_I > \bar{\varepsilon}$ such that the following is true:*

- (i) *If $\varepsilon < \bar{\varepsilon}$, no trade takes place.*
- (ii) *If $\varepsilon \in [\bar{\varepsilon}, \tilde{\varepsilon}_I)$, the quantity exchanged is too high.*
- (iii) *If $\varepsilon > \tilde{\varepsilon}_I$, the quantity exchanged is too low.*

Moreover, if $r > r_0$, where r_0 is defined in the proof, then $\tilde{\varepsilon}_I < \varepsilon_{\text{sup}}$.

Proposition 3 establishes the existence of a unique monetary equilibrium, which is generically inefficient. There are three types of inefficiencies. First, if $\varepsilon < \bar{\varepsilon}$, no goods are exchanged. Second, if $\bar{\varepsilon} \leq \varepsilon < \tilde{\varepsilon}_I$, the quantities exchanged are inefficiently high (prices are too low). Third, if $\varepsilon > \tilde{\varepsilon}_I$, the quantities exchanged are inefficiently low (prices are too high). These inefficiencies occur simultaneously if $r > r_0$, where r_0 is negative if μ is large. They are displayed in Figure 1(a), where the curve labelled q_ε^* displays efficient quantities and the curve labelled q_ε exchanged quantities as functions of ε .

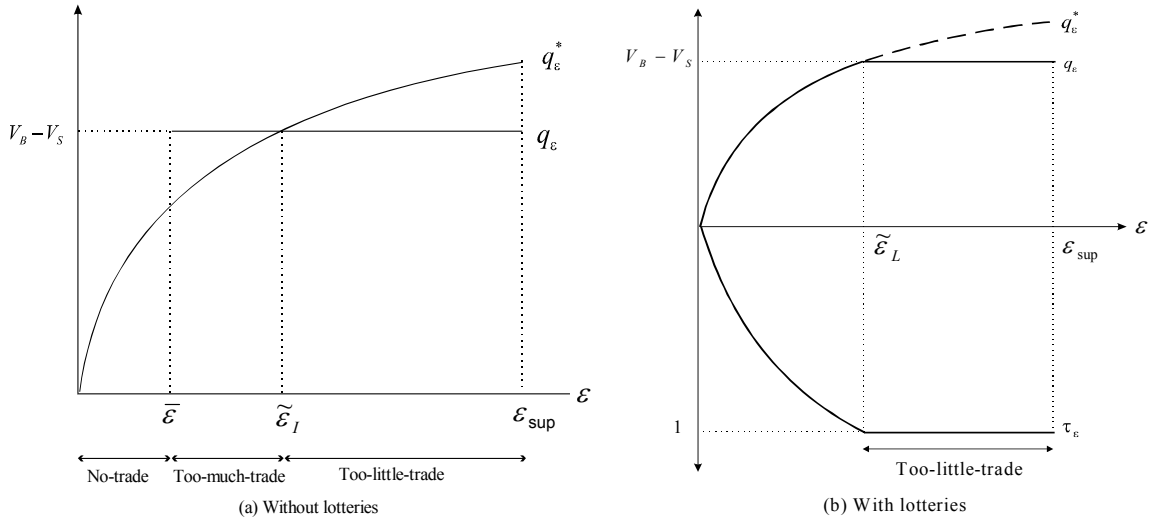


Figure 1. Terms of trade and inefficiencies.

To explain the no-trade and too-much-trade inefficiencies, consider a buyer's consumption decision when the seller produces a variety in a neighborhood of $\bar{\varepsilon}$ (see Figure 1(a)). If $\varepsilon = \bar{\varepsilon}$, the buyer is just indifferent between consuming q units of the good and becoming a seller and remaining a buyer. The seller is also indifferent between producing q units and becoming a buyer and remaining a seller. If ε is slightly below $\bar{\varepsilon}$, no trade takes place, because the bid price of money is smaller than the ask price of money.¹¹ In contrast, if ε is slightly above $\bar{\varepsilon}$,

¹¹The bid price of money is the quantity the seller is willing to produce for one unit of money, and the ask price of money is what the buyer demands to give it up (see Wallace (1997)).

then the bid price of money is larger than its ask price and, because of the buyer-takes-all bargaining protocol, a trade takes place at the bid price. The consumed quantity, however, is inefficiently large because of the buyer's low valuation of the variety.

In the indivisible goods model of Section 3.1, an increase of the inflation rate increases the frequency of trades and welfare by lowering the reservation value $\bar{\varepsilon}$. In the divisible goods model of this subsection, this extensive effect of inflation is also present. In contrast to the indivisible goods model, however, inflation has also an intensive effect, i.e., an increase in the inflation rate reduces the quantities exchanged in each match. In Figure 1, if the inflation rate increases, q is reduced (the intensive effect) and the reservation value $\bar{\varepsilon}$ moves to the left (the extensive effect).

Finally, we want to emphasize that none of these inefficiencies are generated by the assumption of take-it-or-leave-it offers by buyers. Rather, they are a general property of indivisible money models.¹² In particular, it will be shown below that the too-much-trade inefficiency is not a consequence of the high bargaining power of buyers as the take-it-or-leave-it-offer might suggest. Indeed, with lotteries on money this inefficiency vanishes even when the buyer has all the bargaining power.

3.4 Divisible goods with lotteries on money

We now allow agents to use lotteries on money to determine the terms of trade.¹³ We restrict our attention to take-it-or-leave-it offers $(q_\varepsilon, \tau_\varepsilon)$, where q_ε is the amount of goods that the seller produces in an ε -meeting and τ_ε is the probability that the unit of money changes hands. Note that we do not need to introduce a reservation value for the taste index in this model, because if a buyer does not want to trade, he can propose $\tau_\varepsilon = 0$.

With lotteries on money, the expected lifetime utilities of buyers and sellers satisfy

$$rV_B = z(1 - N) \int_0^{\varepsilon_{\sup}} [\varepsilon u(q_\varepsilon) - \tau_\varepsilon (V_B - V_S)] dF(\varepsilon) + \mu (V_S - V_B) \quad (13)$$

$$rV_S = zN \int_0^{\varepsilon_{\sup}} [-q_\varepsilon + \tau_\varepsilon (V_B - V_S)] dF(\varepsilon) + \delta (V_B - V_S) \quad (14)$$

Under buyer-takes-all bargaining, the first term of the right-hand side of (14) is equal to

¹²In Appendix B2, we show that these inefficiencies are also present when the bargaining proceeds according to the generalized Nash bargaining solution. The main difference is that q_ε is not constant but a strictly decreasing function of ε .

¹³We only consider lotteries on money because we want to focus on the consequences of the indivisibility of money without affecting how goods are traded. However, allowing agents to also use lotteries on goods would not change our results at all. Indeed, Berentsen et al. (2000) have shown that in equilibrium goods always change hands with probability 1. Appendix B3 shows that the results presented here also hold if the terms of trade are determined by the generalized Nash bargaining solution.

zero, because the take-it-or-leave-it offers satisfy

$$\tau_\varepsilon (V_B - V_S) = q_\varepsilon \quad \forall \varepsilon \in [0, \varepsilon_{\sup}] \quad (15)$$

For a given realization of the taste index ε , the buyer solves

$$\max_{q_\varepsilon, \tau_\varepsilon} \varepsilon u(q_\varepsilon) - \tau_\varepsilon (V_B - V_S) \quad (16)$$

subject to (15) and

$$\tau_\varepsilon \leq 1 \quad \forall \varepsilon \in [0, \varepsilon_{\sup}] \quad (17)$$

Definition 4 *For the divisible goods and indivisible money model with lotteries on money, a monetary equilibrium is a list $(V_B, V_S, q_\varepsilon, \tau_\varepsilon)$ such that the value functions satisfy (13) and (14) taking the lottery as given; the lottery solves the maximization problem in (16) taking the value functions as given; and $V_B - V_S > 0$.*

As in the model without lotteries, efficiency requires that q_ε satisfy $\varepsilon u'(q_\varepsilon) = 1$ for all $\varepsilon \in [0, \varepsilon_{\sup}]$. Denote efficient quantities by q_ε^* .

Proposition 4 *Consider the model with divisible goods, indivisible money, and lotteries on money. Then, there is a unique monetary equilibrium with $\tilde{\varepsilon}_L > 0$ such that:*

- (i) *If $\varepsilon > \tilde{\varepsilon}_L$, then $q_\varepsilon \in (0, q_\varepsilon^*)$ and $\tau_\varepsilon = 1$.*
- (ii) *If $\varepsilon \leq \tilde{\varepsilon}_L$, then $q_\varepsilon = q_\varepsilon^*$ and $\tau_\varepsilon \leq 1$.*

Furthermore, if $r \leq r_1$, where r_1 is defined in the proof, then $\tilde{\varepsilon}_L \geq \varepsilon_{\sup}$.

The key result is that $0 < q_\varepsilon \leq q_\varepsilon^*$ for all $\varepsilon > 0$. That is, the quantities exchanged are never larger than the efficient quantities, and the frequency of trades is at its efficient level. This result suggests that the no-trade and the too-much-trade inefficiencies are due to the indivisibility of money.

The results of Proposition 4 are displayed in Figure 1(b). In the upper quadrant of Figure 1(b), the dashed curve labelled q_ε^* plots efficient quantities as a function of ε , and the solid curve labelled q_ε displays the quantities that are exchanged in equilibrium. If $\varepsilon < \tilde{\varepsilon}_L$, the two curves merge and the traders exchange efficient quantities. The lower quadrant plots the probability that money changes hands, τ_ε , as a function of the taste index.

With lotteries inflation has only an intensive effect ($\frac{\partial q_\varepsilon}{\partial \mu} < 0$ if $\tilde{\varepsilon} < \varepsilon$). In Figure 1(b), if μ increases, the flat part of the q_ε curve shortens and $\tilde{\varepsilon}_L$ moves to the left. Notice that all trades are efficient if $r \leq r_1$. If $\mu = \delta = 0$, then $r \leq r_1$ is satisfied for r close to 0. Thus, if the traders are patient and inflation is low, they trade efficient quantities in all meetings. In contrast, if μ is large, the too-little-trade inefficiency does not vanish, even when r approaches zero (i.e., r_1 is negative).

Proposition 5 *The measure of trades where the quantities produced are inefficiently low is smaller with lotteries than without, that is, $\tilde{\varepsilon}_L > \tilde{\varepsilon}_I$. Moreover, the value of money ($V_B - V_S$) is higher when lotteries are allowed.*

According to Proposition 5, lotteries reduce the set of matches where the quantities exchanged are inefficiently low. The reason for this result is that money is more valuable with lotteries than without (i.e., $V_B - V_S$ is larger), which implies that sellers are willing to produce more for it.

4 Divisible money

In the indivisible money model, the frequency of trades is too low, because buyers are too choosy. In the divisible goods and indivisible money model, there are inefficiencies associated with no trade, too little trade, and too much trade. What lies behind these inefficiencies?

The analysis of lotteries in Section 3 has suggested that the reason for the no-trade and too-much-trade inefficiencies is the indivisibility of money. To explore this conjecture further, we analyze the same mismatch problem in the divisible money and divisible goods model of Shi (1999).

4.1 The divisible money framework

Most of the description of the environment presented in Section 2 applies to the divisible money environment. The main differences are that time is discrete, and that each household consists of a continuum of members normalized to one who carry out different tasks but regard the household's utility as the common objective. When carrying out these tasks, household members follow the strategy that has been given to them by their households. At the end of each period, they pool their money holdings, which eliminates aggregate uncertainty for households. In the symmetric monetary equilibrium, the distribution of money holdings is degenerate across households. This facilitates the analysis, because we can focus on a representative household.¹⁴ Finally, the utility of the household is defined as the sum of the utilities of its members.¹⁵

¹⁴The large household assumption, extending a similar one in Lucas (1990), avoids difficulties that arise in models with a nondegenerate distribution of money holdings, and so allows for a tractable analysis of inflation. In search models of money, it was first used by Shi (1997, 1999, 2001). See also Berentsen and Rocheteau (2000, 2001). Lagos and Wright (2001) investigate an alternative assumption that yields a degenerate distribution of money holdings in random-matching models of money.

¹⁵We have also introduced the large household in the indivisible money and indivisible goods model of Kiyotaki and Wright (1991, 1993) and the indivisible money and divisible goods model of Trejos and Wright (1995) and Shi (1995). Interestingly, we have found exactly the same reduced form equations as in the

Household members are grouped into money holders (buyers) and producers (sellers). Buyers attempt to exchange money for consumption goods, and sellers attempt to produce goods for money. The fraction of buyers is given by the exogenous constant N . In each period, each household member meets at random a member from another household. Hence, the probability that a seller of type h meets a buyer of type $h - 1$ is zN , and the probability that a buyer of type h meets a seller of type $h + 1$ is $z(1 - N)$.

Although households differ in their preferences and production opportunities, they all consume and produce the same quantities, so that each household can be treated symmetrically. In the following we refer to an arbitrary household as household i . Decision variables of this household are denoted by lowercase letters. Capital letters denote other households' variables, which are taken as given by the representative household i . Because we focus on steady state equilibria, we omit the time index t . Nevertheless, variables corresponding to the next period are indexed by $+1$, and those corresponding to the previous period are indexed by -1 .

The chronology of events within a period is as follows. At the beginning of each period, household i has m units of money, which it divides evenly among its buyers so that each buyer holds m/N units of money in a match. Within a period, no buyer can transfer money to another member of the same household. Then, the household specifies the trading strategies for its members. After this, agents are matched and carry out their exchanges according to the prescribed strategies. After trading, buyers consume the acquired goods, and sellers bring back their receipts of money.¹⁶ At the end of a period, the household receives a lump-sum money transfer τ , which can be negative, and carries the stock m_{+1} to $t + 1$.

The quantity of money in the economy is assumed to grow at the gross growth rate γ . We restrict γ to be larger than the discount factor β .¹⁷ The (indirect) marginal utility of money of the household i is $\omega = \beta V'(m_{+1})$, where $V(m)$ is the steady-state lifetime discounted utility of a household holding m units of money.

As in the previous section, we assume that the terms of trade are determined through take-it-or-leave-it offers by buyers. When matched, household members observe the match type but cannot observe the marginal value of money of their trading partners.¹⁸ As a consequence, households' strategies depend on the match type and on the distribution of their original models (a proof of this claim is available by request).

¹⁶In contrast to Shi (1999), we assume that at the beginning of each period the N buyers are chosen at random among household's members and that each buyer consumes immediately the goods he has acquired in the market.

¹⁷This condition guarantees the existence of a unique steady-state monetary equilibrium.

¹⁸As in Shi (1999), but in contrast to Rauch (2000), buyers' strategies do not depend on the specific characteristics of the sellers they meet; rather they depend on the average characteristics of sellers. For a discussion of this assumption see Berentsen and Rocheteau (2001).

potential bargaining partners' characteristics. In equilibrium, this distribution is degenerate: all households have the same marginal value of money.

A buyer's take-it-or-leave-it offer is a pair $(q_\varepsilon, x_\varepsilon)$, where q_ε is the quantity of goods produced by the seller for x_ε units of money. If the seller accepts the offer, the acquired money x_ε will add to his household's money balances at the beginning of the next period. Because each seller is atomistic, the amount of money obtained by a seller is valued at the marginal utility of money, Ω .¹⁹ The cost associated with this trade is q_ε , and the seller accepts the offer if $x_\varepsilon \Omega \geq q_\varepsilon$. Thus, any optimal offer satisfies

$$x_\varepsilon \Omega = q_\varepsilon \quad \forall \varepsilon \in [0, \varepsilon_{\sup}] \quad (18)$$

Because a buyer cannot exchange more money than he has, the offer $(q_\varepsilon, x_\varepsilon)$ satisfies

$$x_\varepsilon \leq \frac{m}{N} \quad \forall \varepsilon \in [0, \varepsilon_{\sup}] \quad (19)$$

4.2 The program of the household

A household's trading strategy consists of the terms of trade $(q_\varepsilon, x_\varepsilon)$ for all $\varepsilon \in [0, \varepsilon_{\sup}]$, and an acceptance rule for each offer $(Q_\varepsilon, X_\varepsilon)$ from a buyer of another household. Buyers of other households make offers that satisfy a condition similar to (18). Consequently, such offers are accepted by sellers of household i . As in the lottery models of Section 3, there is no need to introduce a reservation value $\bar{\varepsilon}$, because the household can always set $x_\varepsilon = 0$ if he does not want to trade.

For each period, the household chooses m_{+1} and the terms of trade $(q_\varepsilon, x_\varepsilon)$ for all $\varepsilon \in [0, \varepsilon_{\sup}]$ to solve the following dynamic programming problem:

$$V(m) = \max_{q_\varepsilon, x_\varepsilon, m_{+1}} \left\{ zN(1-N) \left(\int_0^{\varepsilon_{\sup}} \varepsilon u(q_\varepsilon) dF(\varepsilon) - \int_0^{\varepsilon_{\sup}} Q_\varepsilon dF(\varepsilon) \right) + \beta V(m_{+1}) \right\} \quad (20)$$

subject to the constraints (18), (19), and

$$m_{+1} - m = \tau + z(1-N)N \int_0^{\varepsilon_{\sup}} X_\varepsilon dF(\varepsilon) - z(1-N)N \int_0^{\varepsilon_{\sup}} x_\varepsilon dF(\varepsilon) \quad (21)$$

¹⁹To see why, suppose that the measure of a member is μ . Then for the household, the value of x additional units of money received by a member is $\beta [V(m_{+1} + x\mu) - V(m_{+1})]$. To express the value of x additional units of money for a member, we must multiply this quantity by the scale factor $\frac{1}{\mu}$. Because members are atomistic, we let $\mu \rightarrow 0$ to get

$$\lim_{\mu \rightarrow 0} \beta \frac{[V(m_{+1} + x\mu) - V(m_{+1})]}{\mu} = x\beta V'(m_{+1}) = x\omega$$

Thus, from the point of view of the household $x\omega$ is a member's indirect utility of receiving x units of money in a match.

The variables taken as given in the above problem are the state variable m and other households' choices. The first term between braces in (20) specifies the consumption utility of the household defined as the sum of utilities of all its members (there is no aggregate uncertainty at the household level). The measure of buyers is N , and the probability of meeting an appropriate seller is $z(1 - N)$, so that the number of single-coincidence meetings involving a buyer in each period is $zN(1 - N)$. Accordingly, aggregate utility is $zN(1 - N) \int_0^{\varepsilon_{\sup}} \varepsilon u(q_\varepsilon) dF(\varepsilon)$. The second term specifies the household's disutility of production.

Equality (21) describes the law of motion of the household's money balances. The first term on the right-hand side specifies the additional currency the household receives each period. The second term specifies sellers' money receipts when selling goods, and the third term specifies buyers' expenses when exchanging money for goods.

Denote by λ_ε the multipliers associated with constraints (19). Note that these constraints are applicable only when buyers are involved in single-coincidence meetings, which occurs with probability $z(1 - N)$. Note further that, according to (18), x_ε can be expressed as a function of q_ε . The first-order conditions and the envelope condition are

$$\varepsilon u'(q_\varepsilon) = \frac{1}{\Omega} (\lambda_\varepsilon + \omega) \quad \forall \varepsilon \in [0, \varepsilon_{\sup}] \quad (22)$$

$$\lambda_\varepsilon \left(x_\varepsilon - \frac{m}{N} \right) = 0 \quad \forall \varepsilon \in [0, \varepsilon_{\sup}] \quad (23)$$

$$\frac{\omega - 1}{\beta} = z(1 - N) \int_0^{\varepsilon_{\sup}} \lambda_\varepsilon dF(\varepsilon) + \omega \quad (24)$$

Equation (22) states that, for a buyer in a desirable match, the marginal utility of consumption must equal the opportunity cost of the amount of money that must be paid to acquire additional goods. To buy another unit of a good, the buyer must give up $\frac{1}{\Omega}$ units of money (see equation (18)). Increasing the monetary payment has two costs to the buyer. He gives up the future value of money ω , and he faces a tighter constraint (19). Together, ω and λ measure the marginal cost of obtaining a larger quantity of goods in exchange for money. Equation (23) is the Kuhn-Tucker condition associated with the multiplier λ_ε . Finally, equation (24) describes the evolution of the marginal value of money. It states that the marginal value of money today, $\frac{\omega - 1}{\beta}$, equals the discounted marginal value of money tomorrow, ω , plus the marginal benefit of relaxing future cash constraints, $z(1 - N) \int_0^{\varepsilon_{\sup}} \lambda_\varepsilon dF(\varepsilon)$.

4.3 Equilibrium

In equilibrium, all households have the same characteristics, which implies that the values for the different variables of household i equal the values of the same variables of all other households. Consequently, capital variables and lowercase variables are equal: $\omega = \Omega$, $m = M$, and $(x_\varepsilon, q_\varepsilon) = (X_\varepsilon, Q_\varepsilon)$ for all $\varepsilon \in [0, \varepsilon_{\sup}]$.

Definition 5 A steady-state monetary equilibrium is a collection $\{(q_\varepsilon), (\lambda_\varepsilon), (x_\varepsilon), m\omega\}$ satisfying equations (18) and (22)–(24), and $m\omega > 0$.

As in Sections 3.3 and 3.4, efficiency requires that the quantities exchanged equalize the marginal utility of the buyer to the marginal cost of the seller. Denote efficient quantities by q_ε^* .

Proposition 6 Consider the divisible goods and divisible money model. Then there is a unique monetary equilibrium with $\omega m > 0$. Furthermore, there is a critical value $\tilde{\varepsilon}_D \leq \varepsilon_{\text{sup}}$ such that the following is true:

- (i) If $\varepsilon > \tilde{\varepsilon}_D$, buyers spend all their money holdings and $q_\varepsilon = \omega \frac{M}{N} < q_\varepsilon^*$.
- (ii) If $\varepsilon \leq \tilde{\varepsilon}_D$, buyers spend only a fraction of their money holdings and $q_\varepsilon = q_\varepsilon^*$.

Moreover, when γ tends to β then $\tilde{\varepsilon}_D$ approaches ε_{sup} .

The key result is that with divisible money we have $q_\varepsilon \in (0, q_\varepsilon^*]$ for all $\varepsilon \in (0, \varepsilon_{\text{sup}}]$: the quantities exchanged are never larger than the efficient quantity, and households consume all varieties. With divisible money, a household simply spends a small amount of money to acquire a small amount of a low-valued variety. Thus, like lotteries, divisible money eliminates the no-trade and the too-much-trade inefficiencies that are present when money is indivisible. The only inefficiency left is the too-little-trade inefficiency that occurs if $\varepsilon > \tilde{\varepsilon}_D$. This inefficiency, however, vanishes under the Friedman rule $\gamma \rightarrow \beta$.²⁰

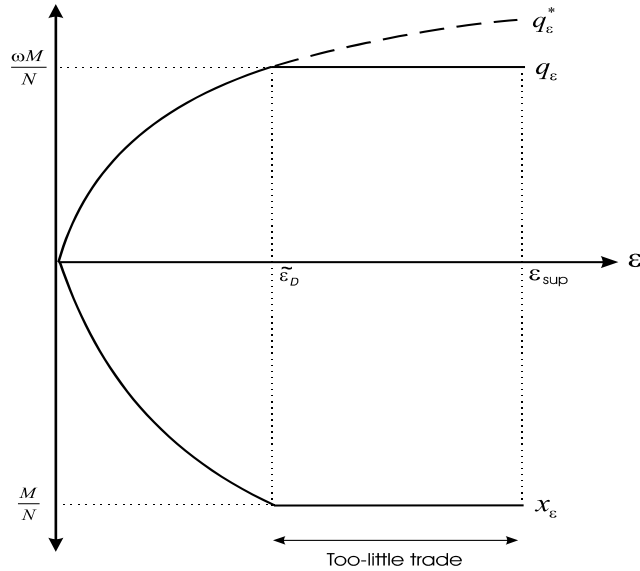


Figure 2: Terms of trade with divisible money.

²⁰The too-little-trade inefficiency may remain under the Friedman rule if buyers do not have all the bargaining power and if sellers can condition their bargaining strategy on the specific level of money holdings of their partner in the match. This issue is discussed in Berentsen and Rocheteau (2001).

Figure 2 illustrates Proposition 6. The upper quadrant displays the efficient quantity q_ε^* and the exchanged quantity q_ε as a function of the taste index ε . If in a match we have $\varepsilon \leq \tilde{\varepsilon}_D$, then $q_\varepsilon = q_\varepsilon^*$, that is, the buyer and the seller produce and consume efficient quantities. If $\varepsilon > \tilde{\varepsilon}_D$, however, then $q_\varepsilon < q_\varepsilon^*$, that is, the quantities exchanged are inefficiently low. The lower quadrant shows the exchanged quantities of money x_ε as a function of ε . If $\varepsilon \leq \tilde{\varepsilon}_D$, then $x_\varepsilon \leq M/N$, that is, the buyer does not spend all his money. If $\varepsilon > \tilde{\varepsilon}_D$, however, $x_\varepsilon = M/N$.

One virtue of our model is that it makes the divisibility of money visible: for low-valued varieties buyers spend only a fraction of their money holdings to acquire an efficient quantity of the good. In contrast, in Shi's (1997, 1999) model buyers always spend their entire money holdings and they always exchange inefficiently low quantities. Moreover, in our framework, in contrast to Shi's (1997, 1999) model, the velocity of money depends on the growth rate of the money supply. In the Appendix A6 we show that the velocity of money is increasing in the rate of inflation. This result arises, because inflation reduces the real value of money (ωM). Consequently, the buyers whose constraints on money holdings are not binding must spend a larger fraction of their money holdings to buy the efficient quantity of goods.

Note also that with divisible money, inflation has no positive “hot potato” effect on welfare, because it cannot increase the frequency of trades. In contrast, inflation is always costly, because it generates a misallocation of resources.²¹ A higher rate of the money supply increases the misallocation, because it reduces the set of meetings where agents produce and exchange efficient quantities. To see this, note that in Figure 2, the horizontal line $\frac{\omega M}{N}$ moves upwards when γ decreases. Consequently, the fraction of inefficient trades decreases. In the limit when γ approaches β , almost all trades are efficient, i.e., $\tilde{\varepsilon}_D$ approaches ε_{sup} .

Note that the model can be reduced to the two following equations that determine q_ε and ω simultaneously:

$$q_\varepsilon = \min \left(q_\varepsilon^*, \frac{\omega M}{N} \right) \quad (25)$$

$$\omega_{-1} = \beta z(1 - N) \int_0^{\varepsilon_{\text{sup}}} \max(\varepsilon u'(q_\varepsilon)\omega, \omega) dF(\varepsilon) + \beta [1 - z(1 - N)] \omega \quad (26)$$

Hence, the model has a very simple structure. Equation (25) states that a buyer consumes the efficient quantity unless his money holdings are too small to compensate the seller for the production cost. Equation (26), which is derived from the envelope condition, can be interpreted as a standard asset pricing equation. For the household, the value of an additional unit of money received at the end of the previous period is ω_{-1} . In the current period, this

²¹With search externalities a departure from the Friedman rule can be a second-best solution (Berentsen, Rocheteau, and Shi (2001)).

unit of money can be either spent or saved. If it is saved, its value from the point of view of the previous period is simply $\beta\omega$. If it is spent in an ε -meeting, the additional utility of consumption is $\varepsilon u'(q_\varepsilon)\omega$. Indeed, from equation (18), one additional unit of money buys ω units of real commodity, where each unit provides $\varepsilon u'(q_\varepsilon)$ additional utility. Accordingly, an additional unit of money is spent if and only if a buyer meets a seller with probability $z(1 - N)$ and if the marginal utility of consumption is larger than the marginal value of money, that is, if $\varepsilon_i u'(q_\varepsilon^b)\omega > \omega$.

Finally, we want to emphasize the similarity of the results of the indivisible money, divisible goods, and lottery model of Section 3 and the divisible money model of this section. This similarity is revealed when comparing Propositions 4 and 6, corresponding respectively to Figures 1b and 2. In both models, there is a threshold $\tilde{\varepsilon}_i$, where $i = L, D$, such that for all $\varepsilon \leq \tilde{\varepsilon}_i$ the quantities exchanged are socially efficient, whereas for all $\varepsilon > \tilde{\varepsilon}_i$ the quantities produced are too low. Moreover, if $\varepsilon > \tilde{\varepsilon}_D$ ($\varepsilon > \tilde{\varepsilon}_L$), buyers spend all their money holdings (they spend the indivisible unit of money with probability one). Also, if $\varepsilon \leq \tilde{\varepsilon}_D$ ($\varepsilon \leq \tilde{\varepsilon}_L$), buyers spend a fraction of their money holdings (they spend the indivisible unit of money with probability $\tau < 1$).

5 Discussion

To explain further the inefficiencies that arise when money is indivisible, and to discuss the role of lotteries to eliminate the too-much-trade and the no-trade inefficiencies, we show graphically how lotteries (or divisible money) affect the Pareto frontier of the bargaining set and the outcome of the bargaining in the divisible goods model.

Consider a match where the taste index of the buyer is ε . When bargaining, the buyer and the seller take V_B and V_S as given. Let \mathcal{S}_B (\mathcal{S}_S) denote the surplus of the buyer (seller), and let $\mathcal{S}^* = \mathcal{S}_B + \mathcal{S}_S$ be the total surplus of the match when the efficient quantity q_ε^* is produced. Note that \mathcal{S}^* is the maximal total surplus that can be attained in this match, and that the $\mathcal{S}^*\mathcal{S}^*$ line in Figure 3 represents the possible divisions of \mathcal{S}^* between the two players.

The Pareto frontiers of the bargaining sets without and with lotteries (\mathcal{P}_I and \mathcal{P}_L) are represented by the solid curves in Figure 3(a) and 3(b), respectively.²² Without lotteries, the Pareto frontier is downward sloping and strictly concave, and it has a unique tangency point with the $\mathcal{S}^*\mathcal{S}^*$ line. With lotteries, the Pareto frontier coincides with the $\mathcal{S}^*\mathcal{S}^*$ line when $\mathcal{S}_S < V_B - V_S - q_\varepsilon^*$, and it is identical with the Pareto frontier of the model without lotteries when $\mathcal{S}_S > V_B - V_S - q_\varepsilon^*$.

²²The expressions for the Pareto frontiers are obtained in Appendix A7.

Figure 3 shows how a lottery on money expands the Pareto frontier of the bargaining set. First, for all $q_\varepsilon > q_\varepsilon^*$, a lottery moves the Pareto frontier upward, i.e., it allows agents to attain a higher total surplus. To see why, note that if $q_\varepsilon > q_\varepsilon^*$, the marginal utility of the buyer is smaller than the marginal cost of the seller. Graphically, the absolute value of the slope of the Pareto frontier is less than one. This overproduction is a social waste, which is removed if we allow lotteries on money, because for all $q_\varepsilon > q_\varepsilon^*$ there is a lottery $\tau < 1$ such that the seller produces q_ε^* and still gets the same surplus as in the model without lottery. This new allocation increases the surplus of the buyer unambiguously. Consequently, lotteries on money permit traders to obtain a Pareto-superior outcome when $q_\varepsilon > q_\varepsilon^*$. Note that the origin of the overproduction in the model without lotteries is that production transfers utility imperfectly. In contrast, when money is *equally* valued by the buyer and the seller, a lottery on money or divisible money transfer utility perfectly. Graphically, the fact that divisible money (or indivisible money with lotteries) transfers utility perfectly is illustrated by the partially linear Pareto frontier in Figure 3b, whereas in Figure 3a the Pareto frontier is strictly concave.²³

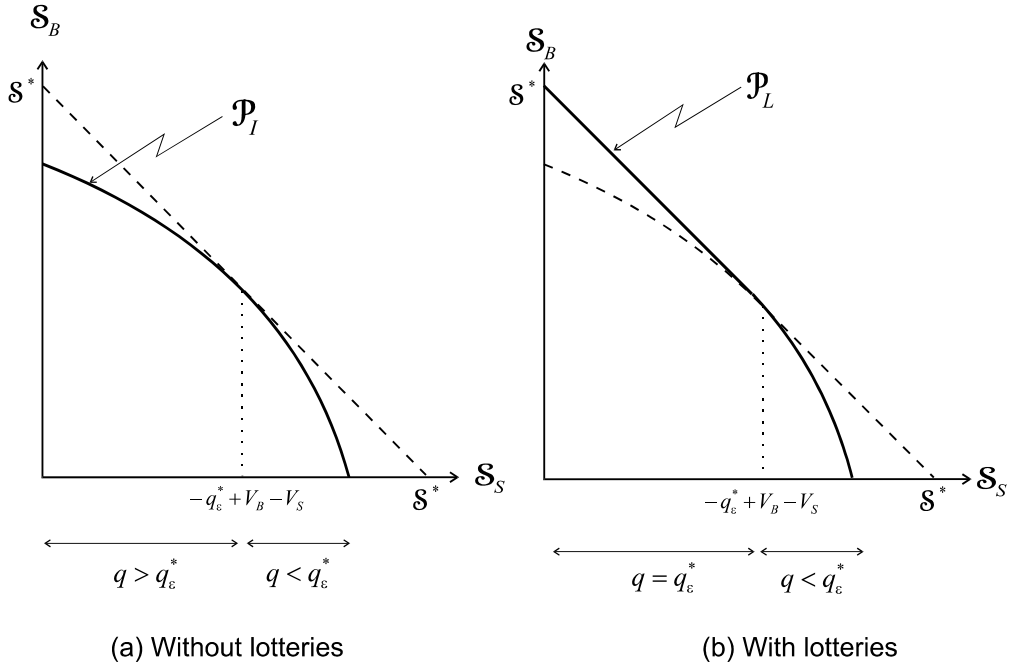


Figure 3: Pareto frontier with and without lotteries.

Second, for all $q_\varepsilon < q_\varepsilon^*$, lotteries do not affect the Pareto frontier. In this case, the marginal utility of the buyer is larger than the marginal cost of the seller. Graphically, the absolute value of the slope of the Pareto frontier is larger than one. To correct this

²³In models where the marginal value of money is not equal for all agents, money cannot transfer utility perfectly (e.g., Berentsen (2000), Rocheteau (2000), and Zhou (1999)).

too-little-trade inefficiency, the buyer must be endowed with more money. Finally, note that increasing the value of money ($V_B - V_S$) moves the point $-q_\varepsilon^* + V_B - V_S$ to the right, which increases the set of points of the Pareto frontier in the model with lotteries (\mathcal{P}_L) that lie on the S^*S^* line.

Figure 3 makes clear that our results in Section 3 do not depend on the bargaining protocol, because the bargaining power of the buyer affects the shape of the Nash product curve, but not the Pareto frontier of the bargaining set.²⁴

In Figure 4 we represent the bargaining solution without lotteries by a circle and the solution with lotteries by a dot when there is the too-much-trade inefficiency (Figure 4a), the too-little-trade inefficiency (Figure 4b), or the no-trade inefficiency (Figure 4c). For each case, under buyer-takes-all bargaining, the bargaining solution is at the point where the Pareto frontier crosses the vertical axes ($\mathcal{S}_S = 0$). In Figure 4a, which is a redraw of Figure 3, lotteries correct the too-little-trade inefficiency. In Figure 4b lotteries cannot solve the too-little-trade inefficiency, because the tangency point of the Pareto frontier with the S^*S^* lies in the second quadrant. In contrast, in Figure 4c lotteries remove the no-trade inefficiency. For this inefficiency, which arises for small values of ε , the Pareto frontier is situated below the horizontal axis, and there is no mutually beneficial trade feasible without lotteries. Introducing lotteries moves the Pareto frontier upward, and the part of the Pareto frontier which is located in the first quadrant coincides with the S^*S^* line.

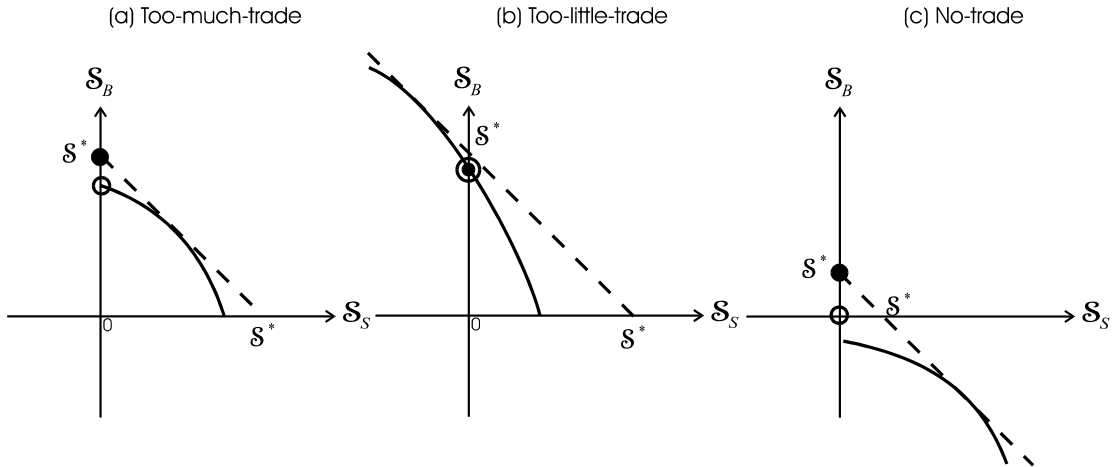


Figure 4: The three inefficiencies.

²⁴Recall that while bargaining the traders take the continuation payoffs V_B and V_S , and consequently the value of money ($V_B - V_S$), as given. If we consider a general change in the bargaining protocol, the shape of the Pareto frontier would change too, because this would affect $V_B - V_S$.

6 Conclusion

Divisible money enables us to distribute “value according to our varying requirements” (Stanley Jevons, 1875). Despite the obvious advantage of having a divisible medium of exchange in real exchanges, in most search models money is an indivisible object.²⁵ The goal of this paper has been to evaluate the consequences of this assumption.

Our main conclusion is that the indivisibility of money generates the *no-trade* and the *too-much-trade* inefficiencies. The reason why indivisible money generates these inefficiencies is intuitive. First, if a buyer’s valuation for a good is very low, then the ask price of money (the quantity that makes the buyer indifferent between trading and not trading) is *larger* than the bid price of money (the quantity that makes the seller indifferent). Consequently, no trade takes place even though it would be socially efficient to trade. This no-trade inefficiency disappears with divisible money (or with lotteries on indivisible money), because if a buyer’s valuation for a good is low, he simply spends a small amount of divisible money (or he delivers the indivisible money with probability less than one) in exchange for a small (and efficient) amount of the good.

Second, if the buyer’s valuation for the good is low but not too low, then the ask price of money is *smaller* than the bid price. Consequently, an exchange takes place. However, because of the buyer’s low valuation for the good, the exchanged quantity is larger than the efficient quantity (this is the too-much-trade inefficiency). Like the no-trade inefficiency, and for the same reason, the too-much-trade inefficiency disappears with divisible money or with lotteries on indivisible money.

Finally, if the buyer’s valuation for the good is large, we observe in all models the too-little-trade inefficiency. The cause of this is that the buyer is constrained by his money holdings. His constraint on money holdings is binding because the purchasing power of money is too low, in turn because agents discount future utilities.

From a methodological perspective, our paper shows that the model with divisible goods, indivisible money, and lotteries on money is qualitatively equivalent to the divisible goods and divisible money model. In either model, the quantities exchanged are efficient for low-valuation goods and are inefficiently low for high-valuation goods. Furthermore, when the quantities exchanged are efficient, the buyer only gives a fraction of his money holdings in the divisible money model, or he delivers his money unit with a probability less than one in the lottery model.

One conclusion of this paper is that an important condition for efficiency in a monetary economy is that the medium of exchange transfers utility perfectly (the exchanged object

²⁵Exceptions are Green and Zhou (1998), Molico (1996), Shi (1997, 1999).

must be equally valued by the two parties). In this sense, our analysis has complemented our paper on money and terms of trade (Berentsen and Rocheteau, 2000), where we demonstrate that barter trades are generically inefficient if the traders have asymmetric preferences for each others' goods. In such an environment, divisible money improves the allocation through its ability to transfer utility perfectly between agents.

APPENDIX A: Proofs

A1. Proof of Proposition 1

Money is accepted by sellers if condition (4) is satisfied, which requires from (2) that $\bar{\varepsilon} \geq \frac{C}{U}$. We will demonstrate that this condition is satisfied if $\frac{C}{U}$ is smaller than some threshold ε_0 .

The individual choice of $\bar{\varepsilon}$. Equations (1), (2), and (3) implicitly define a reaction function $\bar{\varepsilon} = \mathcal{R}(\bar{E})$:

$$(r + \mu + \delta) \bar{\varepsilon} + zN [1 - F(\bar{E})] (\bar{\varepsilon} - C/U) = z(1 - N) \int_{\bar{\varepsilon}}^{\varepsilon_{\sup}} [1 - F(\varepsilon)] d\varepsilon \quad (27)$$

By differentiating (27), we find

$$\mathcal{R}'(\bar{E}) = \frac{zN f(\bar{E}) [\mathcal{R}(\bar{E}) - C/U]}{r + \mu + \delta + zN [1 - F(\bar{E})] + z(1 - N) [1 - F(\mathcal{R}(\bar{E}))]} \quad (28)$$

The critical value ε_0 . The critical value $\varepsilon_0 \in (0, \varepsilon_{\sup})$ is defined as $\varepsilon_0 = \mathcal{R}(\varepsilon_{\sup})$. According to (27), it satisfies

$$(r + \mu + \delta) \varepsilon_0 = z(1 - N) \int_{\varepsilon_0}^{\varepsilon_{\sup}} [1 - F(\varepsilon)] d\varepsilon \quad (29)$$

The left-hand side of (29) is strictly increasing in $\bar{\varepsilon}$, whereas the right-hand side is strictly decreasing for all $\bar{\varepsilon} \in (0, \varepsilon_{\sup})$. Furthermore, $LHS(0) < RHS(0)$ and $LHS(\varepsilon_{\sup}) > RHS(\varepsilon_{\sup})$. Therefore, ε_0 is unique. Moreover, according to (29),

$$\varepsilon_0 \geq \frac{C}{U} \iff z(1 - N) \int_{\frac{C}{U}}^{\varepsilon_{\sup}} [1 - F(\varepsilon)] d\varepsilon \geq (r + \mu + \delta) \frac{C}{U} \quad (30)$$

According to (27),

$$\forall \bar{E} \in [0, \varepsilon_{\sup}], \quad \bar{\varepsilon} = \mathcal{R}(\bar{E}) \geq \frac{C}{U} \iff z(1 - N) \int_{\frac{C}{U}}^{\varepsilon_{\sup}} [1 - F(\varepsilon)] d\varepsilon \geq (r + \mu + \delta) \frac{C}{U} \quad (31)$$

From (30) and (31) we deduce that (see also Figure A1)

$$\forall \bar{E} \in [0, \varepsilon_{\sup}], \quad \varepsilon_0 \geq \frac{C}{U} \iff \mathcal{R}(\bar{E}) \geq \frac{C}{U} \quad (32)$$

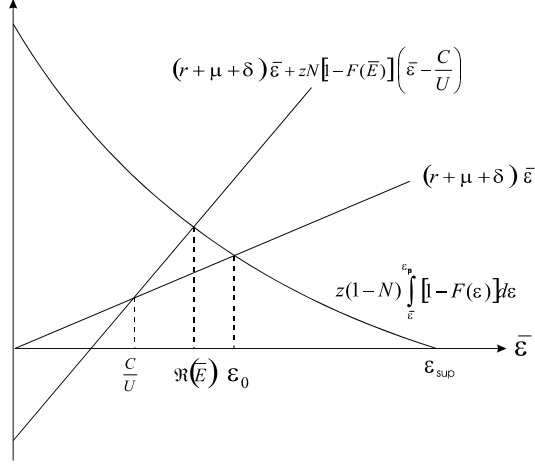


Figure A1.

Existence of a monetary equilibrium. At a symmetric monetary equilibrium, $\bar{\varepsilon} = \mathcal{R}(\bar{E}) \geq C/U$. Thus, condition (32) implies that:

- (i) If $\varepsilon_0 < C/U$, no monetary equilibrium exists.
- (ii) If $\varepsilon_0 \geq C/U$, a monetary equilibrium exists. To see this, note that from (32), if $\varepsilon_0 > C/U$, then $\mathcal{R}(C/U) > C/U$, $\mathcal{R}(\varepsilon_{\sup}) = \varepsilon_0 < \varepsilon_{\sup}$, and from (28) $\mathcal{R}'(\bar{E}) > 0$. Consequently, there is at least one $\bar{E} \in (C/U, \varepsilon_0)$ that satisfies $\bar{E} = \mathcal{R}(\bar{E})$. If $\varepsilon_0 = C/U$, equations (27) and (29) imply that there is a unique symmetric Nash equilibrium with $\bar{E} = \mathcal{R}(\bar{E}) = C/U$.

A2. Proof of Proposition 2

The reservation value $\bar{\varepsilon}$. Inserting equation (7) into equation (8) yields $\bar{\varepsilon}U - C = 0$. Hence, $\bar{\varepsilon} = C/U$.

The lotteries on money τ_ε . Equation (7) implies that

$$\tau_\varepsilon = \tau \equiv C/(V_B - V_S) \quad (33)$$

To derive τ use equations (5), (6), and (8), and integrate by parts to get

$$(r + \mu + \delta)(V_B - V_S) = z(1 - N)U \int_{C/U}^{\varepsilon_{\sup}} [1 - F(\varepsilon)] d\varepsilon \quad (34)$$

Replacing $V_B - V_S$ by its expression given in (34) yields the expression for τ in Proposition 2.

The value functions V_B and V_S . Given $\bar{\varepsilon}$, τ_ε , and $V_B - V_S$, the value functions V_B and V_S are uniquely determined by (5) and (6).

Existence and uniqueness. Because there is a unique $V_B - V_S$ that satisfies (34), if it exists, the monetary equilibrium is unique. For a monetary equilibrium to exist, $V_B - V_S \geq C$, which from (33) implies $\tau \leq 1$. It can be verified from (34) that $V_B - V_S \geq C \Leftrightarrow C/U \leq \varepsilon_0$, where ε_0 satisfies (29).

A3. Proof of Proposition 3

Existence and uniqueness. Equations (9), (10), and (11) yield

$$(r + \mu + \delta)q = z(1 - N) \int_{\bar{\varepsilon}}^{\varepsilon_{\sup}} [\varepsilon u(q) - q] dF(\varepsilon) \quad (35)$$

From (12), (35), and integration by parts we have

$$\left(r + \frac{\mu}{1 - N}\right) \bar{\varepsilon} = z(1 - N) \int_{\bar{\varepsilon}}^{\varepsilon_{\sup}} [1 - F(\varepsilon)] d\varepsilon \quad (36)$$

The left-hand side of (36) is strictly increasing and the right-hand side is strictly decreasing in $\bar{\varepsilon}$ for all $\bar{\varepsilon} < \varepsilon_{\sup}$. Because $RHS(0) > LHS(0) = 0$ and $LHS(\varepsilon_{\sup}) > RHS(\varepsilon_{\sup}) = 0$, a unique reservation value $\bar{\varepsilon} \in (0, \varepsilon_{\sup})$ exists. For a given $\bar{\varepsilon}$, the equilibrium quantity of goods traded in a match is the unique value of q that solves (12). For a given $(\bar{\varepsilon}, q)$, (V_B, V_S) is entirely determined by (9) and (10). Hence, the monetary equilibrium is unique. Note that equations (12) and (36) imply that $\frac{\partial \bar{\varepsilon}}{\partial \mu} < 0$, $\frac{\partial \bar{\varepsilon}}{\partial r} < 0$, $\frac{\partial q}{\partial \mu} < 0$, and $\frac{\partial q}{\partial r} < 0$.

Efficiency. Recall that the efficient quantity q_{ε}^* is an increasing function of the taste index ε . As a consequence, there is a threshold $\tilde{\varepsilon}_I$ for the taste index such that if $\varepsilon > \tilde{\varepsilon}_I$, then $q < q_{\varepsilon}^*$, and if $\varepsilon < \tilde{\varepsilon}_I$, then the opposite is true. The threshold $\tilde{\varepsilon}_I$ satisfies

$$\tilde{\varepsilon}_I u'(q) = 1$$

Using the fact that $\bar{\varepsilon} u(q) = q$ and $\bar{\varepsilon} u'(q_{\bar{\varepsilon}}^*) = 1$, and the concavity of $u(\cdot)$, it can be verified that $q > q_{\bar{\varepsilon}}^*$ and therefore $\tilde{\varepsilon}_I > \bar{\varepsilon}$. Consequently, if $\varepsilon < \bar{\varepsilon}$, no trade takes place; if $\varepsilon \in [\bar{\varepsilon}, \tilde{\varepsilon}_I)$, the quantity exchanged is too high; and if $\varepsilon > \tilde{\varepsilon}_I$, the quantity exchanged is too low.

Condition for $\tilde{\varepsilon}_I < \varepsilon_{\sup}$. To determine the critical value r_0 , note that (12) and (36) yield

$$(r + \mu + \delta) \frac{q}{u(q)} = z(1 - N) \int_{\frac{q}{u(q)}}^{\varepsilon_{\sup}} [1 - F(\varepsilon)] d\varepsilon \quad (37)$$

The left-hand side of (37) is increasing in q , whereas the right-hand side is decreasing in q . If $\tilde{\varepsilon}_I < \varepsilon_{\sup}$, then $q_{\tilde{\varepsilon}_{\sup}}^* > q$, respectively $LHS(q_{\tilde{\varepsilon}_{\sup}}^*) > RHS(q_{\tilde{\varepsilon}_{\sup}}^*)$. Thus,

$$\left(r + \frac{\mu}{1 - N}\right) q_{\tilde{\varepsilon}_{\sup}}^* > z(1 - N) u(q_{\tilde{\varepsilon}_{\sup}}^*) \int_{\frac{q_{\tilde{\varepsilon}_{\sup}}^*}{u(q_{\tilde{\varepsilon}_{\sup}}^*)}}^{\varepsilon_{\sup}} [1 - F(\varepsilon)] d\varepsilon \quad (38)$$

The quantity r_0 is the value of r such that (38) is satisfied with equality.

A4. Proof of Proposition 4

Terms of trade. Equality (15) and inequality (17) imply $q_\varepsilon \leq V_B - V_S$ for all $\varepsilon \in [0, \varepsilon_{\sup}]$. Denote by λ_ε the multiplier associated with this constraint, and rewrite the program (16) to get

$$\max_{q_\varepsilon} \varepsilon u(q_\varepsilon) - q_\varepsilon - \lambda_\varepsilon (V_B - V_S - q_\varepsilon)$$

The first-order and the Kuhn-Tucker conditions are

$$\varepsilon u'(q_\varepsilon) = 1 + \lambda_\varepsilon \quad \forall \varepsilon \in [0, \varepsilon_{\sup}] \quad (39)$$

$$\lambda_\varepsilon (V_B - V_S - q_\varepsilon) = 0 \quad \forall \varepsilon \in [0, \varepsilon_{\sup}] \quad (40)$$

Suppose, first, that inequality (17) is not binding ($\lambda_\varepsilon = 0$). This implies that $q_\varepsilon = q_\varepsilon^* = \tau_\varepsilon (V_B - V_S)$ and $\tau_\varepsilon \leq 1$. Suppose, next, that inequality (17) is binding ($\lambda_\varepsilon > 0$). This implies that $\tau_\varepsilon = 1$ and $q_\varepsilon = V_B - V_S < q_\varepsilon^*$.

The threshold $\tilde{\varepsilon}_L$. From the previous discussion we have $\tau_\varepsilon = \min\left(1, \frac{q_\varepsilon^*}{V_B - V_S}\right)$, where $\frac{q_\varepsilon^*}{V_B - V_S}$ is increasing in ε . Define $\tilde{\varepsilon}_L$ as the smallest ε for which the constraint $\tau_\varepsilon \leq 1$ binds. It satisfies $q_{\tilde{\varepsilon}_L}^* = V_B - V_S$, which can be rewritten as

$$\tilde{\varepsilon}_L = \frac{1}{u'(V_B - V_S)} \quad (41)$$

If $\varepsilon > \tilde{\varepsilon}_L$, then $q_\varepsilon = V_B - V_S < q_\varepsilon^*$ and $\tau_\varepsilon = 1$, and if $\varepsilon \leq \tilde{\varepsilon}_L$, then $q_\varepsilon = q_\varepsilon^*$ and $\tau_\varepsilon < 1$.

Existence and uniqueness. Equations (13), (14), (15), and (41) yield

$$(r + \mu + \delta)(V_B - V_S) = z(1 - N)\Psi \quad (42)$$

$$\text{where } \Psi = \int_0^{\frac{1}{u'(V_B - V_S)}} [\varepsilon u(q_\varepsilon^*) - q_\varepsilon^*] dF(\varepsilon) + \int_{\frac{1}{u'(V_B - V_S)}}^{\varepsilon_{\sup}} [\varepsilon u(V_B - V_S) - (V_B - V_S)] dF(\varepsilon)$$

Note that

$$\begin{aligned} \frac{\partial \Psi}{\partial (V_B - V_S)} &= \int_{\tilde{\varepsilon}_L}^{\varepsilon_{\sup}} [\varepsilon u'(V_B - V_S) - 1] dF(\varepsilon) \geq 0 \\ \frac{\partial^2 \Psi}{\partial (V_B - V_S)^2} &= \int_{\tilde{\varepsilon}_L}^{\varepsilon_{\sup}} [\varepsilon u''(V_B - V_S)] dF(\varepsilon) \leq 0 \end{aligned}$$

Equation (42) determines a unique $V_B - V_S$. Indeed, the left-hand side of (42) is linear in $V_B - V_S$. The right-hand side of (42) is constant if $\varepsilon_{\sup} \leq \tilde{\varepsilon}_L$, and strictly increasing and strictly concave if $\varepsilon_{\sup} > \tilde{\varepsilon}_L$. Moreover, $LHS(0) = RHS(0) = 0$ and $LHS'(0) = r + \mu + \delta < RHS'(0) = +\infty$. Consequently, there is a unique strictly positive value of $V_B - V_S$ that satisfies (42).

Condition for $\tilde{\varepsilon}_L \geq \varepsilon_{\sup}$. We have $\tilde{\varepsilon}_L \geq \varepsilon_{\sup} \Leftrightarrow q_{\varepsilon_{\sup}}^* \leq V_B - V_S$, which from (42) is also equivalent to

$$\left(r + \frac{\mu}{1-N}\right) q_{\varepsilon_{\sup}}^* \leq z(1-N) \int_0^{\varepsilon_{\sup}} [\varepsilon u(q_{\varepsilon}^*) - q_{\varepsilon}^*] dF(\varepsilon) \quad (43)$$

Denote by r_1 the value of r that satisfies (43) with equality. Then $\tilde{\varepsilon}_L \geq \varepsilon_{\sup} \Leftrightarrow r \leq r_1$.

A5. Proof of Proposition 5

The thresholds $\tilde{\varepsilon}_i$, $i \in \{I, L\}$, satisfy $\tilde{\varepsilon}_i u'(V_B - V_S) = 1$. Thus, $\tilde{\varepsilon}_L > \tilde{\varepsilon}_I$ iff $V_B - V_S$ is larger with lotteries than without. From (12) and (35), the equation that determines $V_B - V_S$ in the model without lotteries is

$$(r + \mu + \delta)(V_B - V_S) = z(1-N) \int_{\frac{V_B - V_S}{u(V_B - V_S)}}^{\varepsilon_{\sup}} [\varepsilon u(V_B - V_S) - (V_B - V_S)] dF(\varepsilon) \quad (44)$$

With lotteries $V_B - V_S$ is determined by (42). From the fact that $\varepsilon u(q_{\varepsilon}^*) - q_{\varepsilon}^* > \varepsilon u(V_B - V_S) - (V_B - V_S)$ for all $\varepsilon \neq \tilde{\varepsilon}$, the RHS of (42) is strictly larger than the RHS of (44). Consequently, $V_B - V_S$ is strictly larger with lotteries than without.

A6. Proof of Proposition 6

Terms of trade. According to (22), if the constraint (19) is not binding ($\lambda_{\varepsilon} = 0$), then $q_{\varepsilon} = q_{\varepsilon}^*$. From (18), the monetary transfer equals $x_{\varepsilon} = q_{\varepsilon}^*/\omega$. In contrast, if (19) is binding ($\lambda_{\varepsilon} > 0$), then, from (18), $q_{\varepsilon} = \frac{\omega M}{N}$ and $x_{\varepsilon} = \frac{M}{N}$. Thus

$$q_{\varepsilon} = \min\left(\frac{\omega M}{N}, q_{\varepsilon}^*\right) \quad (45)$$

The threshold $\tilde{\varepsilon}_D$. The threshold $\tilde{\varepsilon}_D$ is the smallest ε for which the constraint $x_{\varepsilon} \leq \frac{M}{N}$ binds. Hence, $\tilde{\varepsilon}_D$ satisfies $\frac{\omega M}{N} = q_{\varepsilon_D}^*$, so that

$$\tilde{\varepsilon}_D u'\left(\frac{\omega M}{N}\right) = 1 \quad (46)$$

Existence and uniqueness. The model has a recursive structure. For a given $\tilde{\varepsilon}_D$, (46) determines ωM . Given ωM , (45) determines q_{ε} . Given q_{ε} , (18) determines x_{ε} and (22) determines λ_{ε} . Furthermore, for a monetary equilibrium to exist, we must have $\omega > 0$, which implies from (46) that $\tilde{\varepsilon}_D > 0$. In the following, we will demonstrate that there exists a unique $\tilde{\varepsilon}_D > 0$.

By substituting for λ_ε its expression given in (22), equation (24) becomes

$$\frac{\omega_{-1}}{\omega} = \beta \left\{ z(1-N) \int_{\tilde{\varepsilon}_D}^{\varepsilon_{\sup}} [\varepsilon u'(q_\varepsilon) - 1] dF(\varepsilon) + 1 \right\} \quad (47)$$

By multiplying each side of (47) by $\tilde{\varepsilon}_D$, from (46), we have

$$\left(\frac{\omega_{-1}}{\omega} - \beta \right) \tilde{\varepsilon}_D = \beta z(1-N) \int_{\tilde{\varepsilon}_D}^{\varepsilon_{\sup}} [\varepsilon - \tilde{\varepsilon}_D] dF(\varepsilon) \quad (48)$$

In the steady state, the quantity ωm is stationary. Therefore, $\frac{m}{m_{-1}} = \frac{\omega_{-1}}{\omega} = \gamma$. Integrating by parts, and taking into account that $\frac{\omega_{-1}}{\omega} = \gamma$, we obtain the equation that determines $\tilde{\varepsilon}_D$ in the steady state:

$$(\gamma - \beta) \tilde{\varepsilon}_D = \beta z(1-N) \int_{\tilde{\varepsilon}_D}^{\varepsilon_{\sup}} [1 - F(\varepsilon)] d\varepsilon \quad (49)$$

The left-hand side of (49) is strictly increasing in $\tilde{\varepsilon}_D$, whereas the right-hand side is strictly decreasing in $\tilde{\varepsilon}_D$. Furthermore, $LHS(0) = 0 < RHS(0)$ and $LHS(\varepsilon_{\sup}) > RHS(\varepsilon_{\sup}) = 0$. Consequently, there is a unique $\tilde{\varepsilon}_D \in (0, \varepsilon_{\sup})$ that satisfies (49). Finally, equation (49) implies that when γ approaches β , $\tilde{\varepsilon}_D$ approaches ε_{\sup} .

Velocity of money. The velocity of money is defined as the ratio of the money transfers in one period divided by the money supply, i.e.,

$$\Lambda = \frac{z(1-N)N \int_0^{\varepsilon_{\sup}} x_\varepsilon dF(\varepsilon)}{M} \quad (50)$$

Proposition 6 and (18) imply that $x_\varepsilon = q_\varepsilon^*$ if $\varepsilon \leq \tilde{\varepsilon}_D$ and $x_\varepsilon = \frac{M}{N}$ if $\varepsilon > \tilde{\varepsilon}_D$. Consequently, (50) can be rewritten as

$$\Lambda = \frac{z(1-N)N}{M\Omega} \int_0^{\tilde{\varepsilon}_D} q_\varepsilon^* dF(\varepsilon) + z(1-N) \int_{\tilde{\varepsilon}_D}^{\varepsilon_{\sup}} dF(\varepsilon), \quad (51)$$

where $\tilde{\varepsilon}_D$ is defined by (46). An increase in the rate of inflation reduces $M\Omega$, and from (51) increases Λ .

A7. Pareto frontier of the bargaining set in the indivisible money, divisible goods model

We consider a meeting between a buyer and a seller, where the taste index of the buyer is ε . Let \mathcal{S}_B (\mathcal{S}_S) denote the surplus of the buyer (seller) in the match. When bargaining, the buyer and the seller take V_B and V_S , and accordingly the value of money ($V_B - V_S$), as given.

Without lotteries. Without lotteries, the surpluses satisfy

$$\mathcal{S}_B = \varepsilon u(q) - (V_B - V_S) \quad (52)$$

$$\mathcal{S}_S = -q + (V_B - V_S) \quad (53)$$

where q is the quantity produced by the seller for the buyer. The Pareto frontier is derived by solving the following maximization problem:

$$\max_q \mathcal{S}_B = \varepsilon u(q) - (V_B - V_S) \quad \text{s.t. (53)}$$

Because there is one variable to be chosen and one constraint, the value of q is directly given by (53). From (52) and (53), the equation for the Pareto frontier in the model without lotteries is given by

$$\mathcal{S}_B = \varepsilon u(V_B - V_S - \mathcal{S}_S) - (V_B - V_S) \quad (54)$$

The Pareto frontier is downward sloping and strictly concave, and at $\mathcal{S}_S = V_B - V_S - q_\varepsilon^*$ its slope is -1 . At $\mathcal{S}_S = 0$ the buyer's surplus equals $\mathcal{S}_B = \varepsilon u(V_B - V_S) - (V_B - V_S)$. This quantity can be negative if ε is low. In this case, there will be no trade.

With lotteries. With lotteries, the surpluses satisfy

$$\mathcal{S}_B = \varepsilon u(q) - \tau(V_B - V_S) \quad (55)$$

$$\mathcal{S}_S = -q + \tau(V_B - V_S) \quad (56)$$

where τ is the probability that the indivisible unit of money changes hands.

The Pareto frontier is derived by solving the following maximization problem:

$$\max_{q, \tau} \mathcal{S}_B = \varepsilon u(q) - \tau(V_B - V_S) \quad \text{s.t. (56) and } 0 \leq \tau \leq 1$$

Assume first that an interior solution to this program exists. Then we have

$$1 = \varepsilon u'(q) \quad (57)$$

$$\tau = \frac{\mathcal{S}_S + q}{V_B - V_S} \quad (58)$$

and the equation for the Pareto frontier satisfies

$$\mathcal{S}_B + \mathcal{S}_S = \mathcal{S}^* \equiv \varepsilon u(q_\varepsilon^*) - q_\varepsilon^*$$

where q_ε^* satisfies (57).

Assume next that no interior solution exists. The constraint $\tau \leq 1$ is binding iff the value of τ given by (58) is larger than 1 (i.e. $\mathcal{S}_S \geq V_B - V_S - q_\varepsilon^*$). Accordingly, the equation of the Pareto frontier is given by (54).

APPENDIX B: Generalization of the terms of trade

In Appendix B we show that the claims we make in the text are also true for generalized Nash bargaining.

B1. Indivisible goods and indivisible money with lotteries

In this appendix, we show that lotteries on money remove the no-trade inefficiency even when the terms of trade are determined by the generalized Nash solution.²⁶ We do not consider the conditions for the existence and uniqueness of an equilibrium. We take as granted that $V_B - V_S \geq C$, which is a necessary and sufficient condition for a monetary equilibrium.

Let $\theta \in (0, 1)$ be the buyers' bargaining power, and τ_ε the probability that money changes hands in an ε -meeting. The outcome of a negotiation between a buyer and a seller is a $\tau_\varepsilon \in [0, 1]$ that maximizes the following Nash product:

$$\tau_\varepsilon = \arg \max [\varepsilon U - \tau_\varepsilon (V_B - V_S)]^\theta [-C + \tau_\varepsilon (V_B - V_S)]^{1-\theta} \quad (59)$$

$$\text{s.t. } 0 \leq \tau_\varepsilon \leq 1 \quad (60)$$

$$\varepsilon U \geq \tau_\varepsilon (V_B - V_S) \quad (61)$$

$$C \leq \tau_\varepsilon (V_B - V_S) \quad (62)$$

A necessary condition for (61) and (62) to be satisfied is that $\varepsilon U \geq C$. The first-order conditions imply that the optimal τ_ε satisfies

$$\tau_\varepsilon = \begin{cases} \frac{(1-\theta)\varepsilon U + \theta C}{V_B - V_S} & \text{if } \frac{C}{U} \leq \varepsilon \leq \frac{V_B - V_S - \theta C}{(1-\theta)U} \\ 1 & \text{if } \varepsilon > \frac{V_B - V_S - \theta C}{(1-\theta)U} \end{cases} \quad (63)$$

The participation constraints (61) and (62) are satisfied for all $\varepsilon \geq \frac{C}{U}$. Indeed, for all $\varepsilon \in \left[\frac{C}{U}, \frac{V_B - V_S - \theta C}{(1-\theta)U} \right]$ we have

$$\begin{aligned} \varepsilon U - \tau_\varepsilon (V_B - V_S) &= \theta (\varepsilon U - C) \geq 0 \\ -C + \tau_\varepsilon (V_B - V_S) &= (1 - \theta) (\varepsilon U - C) \geq 0 \end{aligned}$$

If $\varepsilon > \frac{V_B - V_S - \theta C}{(1-\theta)U}$, the participation constraints are satisfied, because $V_B - V_S \geq C$.

²⁶We only allow lotteries on money, because we want to focus on the welfare consequences of indivisible money. When agents use lotteries on both goods and money, the main difference is that the good changes hands with probability less than one if ε is large and if the sellers have sufficient bargaining power. The fact that agents trade goods with probability less than one corresponds to what we call the *too-little-trade* inefficiency in the divisible goods, indivisible money, and lottery model of Section 3.3.

The surplus in the match of the buyer is $\theta(\varepsilon U - C)$ for all $\varepsilon \leq \frac{V_B - V_S - \theta C}{(1-\theta)U}$ and $\varepsilon U - (V_B - V_S)$ for all $\varepsilon > \frac{V_B - V_S - \theta C}{(1-\theta)U}$. This surplus is increasing in ε . Therefore, buyers choose a reservation value $\bar{\varepsilon}$ for the taste index below which they refuse to trade. The reservation value $\bar{\varepsilon}$ is such that buyers are indifferent between trading and not trading. Consequently, the reservation value is $\bar{\varepsilon} = \frac{C}{U}$, which coincides with the value that a social planner would choose.

B2. Divisible goods and indivisible money

This appendix shows that the no-trade, the too-much-trade, and the too-little-trade inefficiencies that we found in Section 3.3 are also present when the terms of trade are determined through generalized Nash bargaining. We will neither demonstrate the existence of a monetary equilibrium nor discuss its uniqueness. We will assume that $V_B - V_S > 0$, which is a necessary condition for the existence of a monetary equilibrium.

The bargaining. The bargaining power of buyers is $\theta \in (0, 1)$, and the outcome of the bargaining is given by the generalized Nash solution. During the negotiation the bargainers continue to meet other traders. Consequently, the *threat points* of buyers and sellers are V_B and V_S , respectively (see Trejos and Wright, 1995). Denote by q_ε the solution to the general Nash bargaining game in a ε -meeting. Then

$$q_\varepsilon = \arg \max_q [\varepsilon u(q) - (V_B - V_S)]^\theta [-q + V_B - V_S]^{1-\theta}$$

The first-order condition is

$$\theta \varepsilon u'(q_\varepsilon) (-q_\varepsilon + V_B - V_S) = (1 - \theta) [\varepsilon u(q_\varepsilon) - (V_B - V_S)] \quad (64)$$

From (64), we can show that:

$$\frac{\partial q_\varepsilon}{\partial \varepsilon} = \frac{(1 - \theta) (V_B - V_S) / \varepsilon^2}{\theta u''(q_\varepsilon) S_S - u'(q_\varepsilon)} < 0$$

where $S_S = -q_\varepsilon + V_B - V_S$ is the seller surplus. Note that the participation constraint of the seller requires $S_S \geq 0$.

The reservation value $\bar{\varepsilon}$. From $\frac{\partial q_\varepsilon}{\partial \varepsilon} < 0$ we deduce that the LHS of (64) is increasing in ε . As a consequence, the surplus of the buyer, $S_B = \varepsilon u(q_\varepsilon) - (V_B - V_S)$, is also increasing in ε . The reservation property holds, which implies that there is a reservation value $\bar{\varepsilon}$ satisfying

$$\bar{\varepsilon} u(q_{\bar{\varepsilon}}) = V_B - V_S \quad (65)$$

The inefficiencies (see Figure B2). According to (65), if $V_B - V_S > 0$, then both $\bar{\varepsilon} > 0$ and $q_{\bar{\varepsilon}} > 0$. Consequently, the no-trade inefficiency remains. The concavity of the utility function implies that the too-much-trade inefficiency is also always present. To see this, note first that equations (64) and (65) imply that the buyer surplus and the seller surplus are zero when $\varepsilon = \bar{\varepsilon}$, which implies that $q_{\bar{\varepsilon}}$ satisfies $\bar{\varepsilon}u(q_{\bar{\varepsilon}}) = q_{\bar{\varepsilon}}$. Note next that $q_{\bar{\varepsilon}}^*$ satisfies $\bar{\varepsilon}u'(q_{\bar{\varepsilon}}^*) = 1$. By combining these two equations we find $u'(q_{\bar{\varepsilon}}^*) = \frac{u(q_{\bar{\varepsilon}})}{q_{\bar{\varepsilon}}}$, which can only be satisfied if $q_{\bar{\varepsilon}} > q_{\bar{\varepsilon}}^*$. Finally, because q_{ε} is decreasing in ε and q_{ε}^* is increasing in ε , there exists a threshold $\tilde{\varepsilon}_I$ such that $q_{\tilde{\varepsilon}_I} = q_{\tilde{\varepsilon}_I}^*$. The too-little-trade inefficiency is present under generalized Nash bargaining if $\tilde{\varepsilon}_I < \varepsilon_{\text{sup}}$.

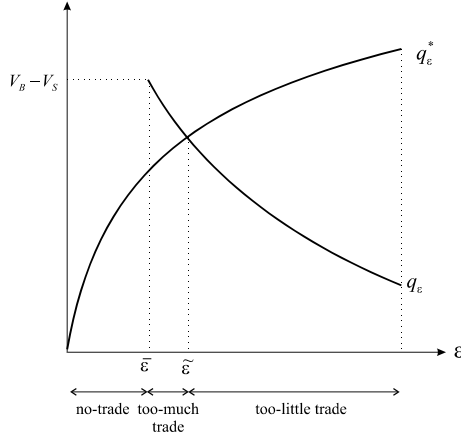


Figure B2.

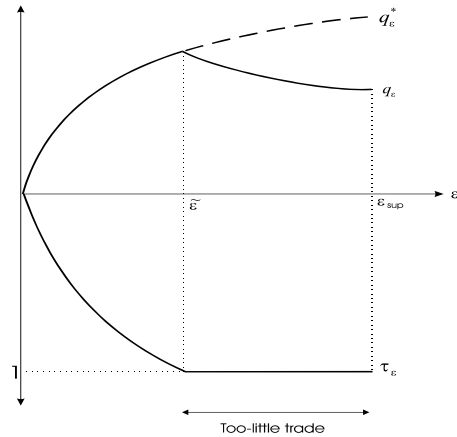


Figure B3.

B3. Divisible goods and indivisible money with lotteries

We will show that with lotteries on money the only inefficiency that remains is the too-little-trade inefficiency. We will also show under which condition this inefficiency vanishes.

The bargaining. The terms of trade in an ε -meeting maximize the Nash product:

$$(q_{\varepsilon}, \tau_{\varepsilon}) = \arg \max_{q, \tau} [\varepsilon u(q) - \tau (V_B - V_S)]^{\theta} [-q + \tau (V_B - V_S)]^{1-\theta} \quad \text{s.t. } \tau_{\varepsilon} \leq 1 \quad (66)$$

If the constraint on τ_{ε} is not binding, then

$$q_{\varepsilon} = q_{\varepsilon}^* \text{ and } \tau_{\varepsilon} = \frac{(1 - \theta) \varepsilon u(q_{\varepsilon}^*) + \theta q_{\varepsilon}^*}{V_B - V_S}$$

Hence, the constraint on τ_{ε} is binding if and only if

$$(1 - \theta) \varepsilon u(q_{\varepsilon}^*) + \theta q_{\varepsilon}^* > V_B - V_S \quad (67)$$

The LHS of (67) is increasing in ε . Consequently, there exists a $\tilde{\varepsilon}_L$ satisfying

$$(1 - \theta) \tilde{\varepsilon}_L u(q_{\tilde{\varepsilon}_L}^*) + \theta q_{\tilde{\varepsilon}_L}^* = V_B - V_S$$

If the constraint $\tau_\varepsilon \leq 1$ is binding, q_ε satisfies (64) and $\tau_\varepsilon = 1$. Equation (64) implies that q_ε is a decreasing function of ε . Because q_ε^* is strictly increasing in ε , we have $q_\varepsilon < q_\varepsilon^*$ for all $\varepsilon > \tilde{\varepsilon}_L$.

The inefficiencies (see Figure B3). For all $\varepsilon \leq \tilde{\varepsilon}_L$ we have $q_\varepsilon = q_\varepsilon^*$ and $\tau_\varepsilon \leq 1$, and for all $\varepsilon > \tilde{\varepsilon}_L$, $q_\varepsilon < q_\varepsilon^*$ and $\tau_\varepsilon = 1$. This confirms that with lotteries the only inefficiency left is the too-little-trade inefficiency.

The condition for efficiency. Traders exchange efficient quantities in all meetings if $\tilde{\varepsilon}_L \geq \varepsilon_{\text{sup}}$. This condition can be rewritten as follows:

$$(1 - \theta) \varepsilon_{\text{sup}} u(q_{\varepsilon_{\text{sup}}}^*) + \theta q_{\varepsilon_{\text{sup}}}^* < V_B - V_S \quad (68)$$

If $\tilde{\varepsilon}_L \geq \varepsilon_{\text{sup}}$, then $q_\varepsilon = q_\varepsilon^*$ and $\tau_\varepsilon (V_B - V_S) = (1 - \theta) \varepsilon u(q_\varepsilon^*) + \theta q_\varepsilon^*$ for all ε . Consequently, the value functions of buyers and sellers satisfy

$$\begin{aligned} rV_B &= z(1 - N) \theta \int_0^{\varepsilon_{\text{sup}}} [\varepsilon u(q_\varepsilon^*) - q_\varepsilon^*] dF(\varepsilon) - \mu (V_B - V_S) \\ rV_S &= zN(1 - \theta) \int_0^{\varepsilon_{\text{sup}}} [\varepsilon u(q_\varepsilon^*) - q_\varepsilon^*] dF(\varepsilon) + \delta (V_B - V_S) \end{aligned}$$

Consequently,

$$(r + \mu + \delta) (V_B - V_S) = z(\theta - N) \int_0^{\varepsilon_{\text{sup}}} [\varepsilon u(q_\varepsilon^*) - q_\varepsilon^*] dF(\varepsilon)$$

The condition (68) yields

$$\frac{z(\theta - N) \int_0^{\varepsilon_{\text{sup}}} [\varepsilon u(q_\varepsilon^*) - q_\varepsilon^*] dF(\varepsilon)}{r + \mu + \delta} > (1 - \theta) \varepsilon_{\text{sup}} u(q_{\varepsilon_{\text{sup}}}^*) + \theta q_{\varepsilon_{\text{sup}}}^*$$

First, a necessary condition for efficiency is that $\theta > N$. This implies that the too-little-trade inefficiency does not vanish if the bargaining power of buyers is less than the fraction of buyers in the economy. Second, in the absence of inflation ($\mu = \delta = 0$) and if $\theta > N$, the efficiency condition is satisfied for r close to zero. If agents become very patient and if the bargaining power of buyers is sufficiently high, then the too-little-trade inefficiency vanishes.

B4. Divisible goods and divisible money

The game. The terms of trades are determined through bargaining games with alternating offers (see Shi, 1999, 2001).²⁷ We consider the bargaining between agent i of household i and an agent j from any other household. Each period is divided into an infinite number of

²⁷See Section 4 for additional details of the bargaining protocol.

subperiods of length Δ . If, in a given subperiod, it is agent i 's turn to make an offer and agent j rejects the offer, in the following subperiod it is agent j 's turn to make a counteroffer. If an offer by a buyer is refused, the negotiation breaks down with probability $\theta\Delta$ ($\theta \in (0, 1)$). If an offer by a seller is refused, the negotiation breaks down with probability $(1 - \theta)\Delta$. We will see that the parameter θ can be interpreted as the bargaining power of buyers. The possibility of an exogenous breakdown of the negotiation gives an incentive to traders to agree immediately.

In the alternating offer game, offers and counteroffers converge to the same limiting proposal when Δ goes to zero. Consequently, the first-mover advantage vanishes when Δ goes to zero.²⁸ Because of this and because, as we will see, it facilitates the derivation of the envelope condition, we let members of household i make the first offer in all meetings. In the symmetric equilibrium all households have the same characteristics. Consequently, the first offers of household i are always accepted. Moreover, because the length of time between two consecutive offers is infinitesimal, these first offers are exactly equal to the counteroffers that household i 's bargaining partners make.

The offers. The bargaining strategies are determined at the level of the household. We assume that these bargaining strategies depend on the specific taste index of the buyer in the match, ε , but they do not depend on the specific level of money holdings or the specific marginal value of money of the buyer. Nonetheless, the bargaining strategies depends on the distribution of money holdings of the buyers and on the distribution of their marginal utilities of money. These two distributions, however, are degenerate in equilibrium.²⁹ Assume first that agent i of household i is the buyer, and that in a given subperiod it is his turn to make an offer. The buyer, following the strategy given to him by household i , proposes that the seller produces q_ε^b units of output in exchange for x_ε^b units of money if the quality of the match is ε . There are two types of constraints that the buyer's household i must take into account. First, the proposed amount of money cannot exceed the buyer's money holdings, i.e.,

$$x_\varepsilon^b \leq \frac{m}{n} \tag{69}$$

²⁸This is a standard argument in the bargaining literature. See, for example, Muthoo (1999, Chapter 3), Osborne and Rubinstein (1990, Chapter 3).

²⁹If sellers can condition their bargaining strategies on the specific level of money holdings of their partner in the match, then, because of the holdup problem, the too-little-trade inefficiency does not vanish under the Friedman rule. Nonetheless, it is feasible to reduce the growth rate of the money supply below what is prescribed by the Friedman rule to eradicate the too-little-trade inefficiency. See Berentsen and Rocheteau (2001) on this point.

This constraint must be satisfied because trade is decentralized and so, during a match, each buyer is separated from other members of the household. The second constraint on the offer is that it must give the partner (a seller from another household) a surplus that is greater than or equal to the partner's reservation surplus. The seller's household obtains the surplus $\Omega x_\varepsilon^b - q_\varepsilon^b$ by accepting the offer, where Ωx_ε^b is the value of the amount of money x_ε^b to the seller's household, and q_ε^b is the production cost. Denote by D_ε^s the expected surplus of the seller if he rejects the offer (the reservation surplus): it is taken as given by the household of the buyer. Note that because the buyer and the seller have opposite interests, an optimal offer must make the seller indifferent between accepting and rejecting. Consequently, the buyer offer $(q_\varepsilon^b, x_\varepsilon^b)$ satisfies

$$-q_\varepsilon^b + x_\varepsilon^b \Omega = D_\varepsilon^s \quad (70)$$

To derive the reservation surplus D_ε^s , note that if the seller rejects the offer (but stays in the game), the game passes into the next round without breakdown with probability $1 - \theta\Delta$, in which the seller proposes (Q^s, X^s) . Taking into account the breakdown probability, the reservation surplus of the seller is

$$D_\varepsilon^s = (1 - \theta\Delta) [-Q_\varepsilon^s + X_\varepsilon^s \Omega] \quad (71)$$

Assume now that agent i is the seller. Then the seller offer $(q_\varepsilon^s, x_\varepsilon^s)$ satisfies

$$x_\varepsilon^s \leq \frac{M}{N} \quad (72)$$

$$\varepsilon u(q_\varepsilon^s) - x_\varepsilon^s \Omega = D_\varepsilon^b \quad (73)$$

where the buyer's reservation surplus D_ε^b is

$$D_\varepsilon^b = (1 - (1 - \theta)\Delta) [\varepsilon u(Q_\varepsilon^b) - X_\varepsilon^b \Omega] \quad (74)$$

The program of the household. For each period, the household chooses $(m_{+1}, q_\varepsilon^b, x_\varepsilon^b, q_\varepsilon^s, x_\varepsilon^s)$ to solve the following dynamic programming problem:

$$V(m) = \max_{m_{+1}, q_\varepsilon^b, x_\varepsilon^b, q_\varepsilon^s, x_\varepsilon^s} \left\{ zN(1 - N) \left(\int_0^{\varepsilon_{\sup}} \varepsilon u(q_\varepsilon^b) dF(\varepsilon) - \int_0^{\varepsilon_{\sup}} q_\varepsilon^s dF(\varepsilon) \right) + \beta V(m_{+1}) \right\} \quad (75)$$

s.t. (70), (69), (73), (72), and

$$m_{+1} - m = \tau + z(1 - N)N \int_0^{\varepsilon_{\sup}} x_\varepsilon^s dF(\varepsilon) - z(1 - N)N \int_0^{\varepsilon_{\sup}} x_\varepsilon^b dF(\varepsilon) \quad (76)$$

where equation (76) describes the law of motion of money holdings.

Denote by λ_ε (π_ε) the multipliers that are associated with constraints (69) ((72)). Then the first-order conditions and the envelope condition are as follows:

$$\varepsilon u'(q_\varepsilon^b) = \frac{\lambda_\varepsilon + \omega}{\Omega} \quad \forall \varepsilon \in [0, \varepsilon_{\text{sup}}] \quad (77)$$

$$\varepsilon u'(q_\varepsilon^s) = \frac{\Omega}{\omega - \pi_\varepsilon} \quad \forall \varepsilon \in [0, \varepsilon_{\text{sup}}] \quad (78)$$

$$\lambda_\varepsilon \left(\frac{m}{N} - x_\varepsilon^b \right) = 0 \quad \forall \varepsilon \in [0, \varepsilon_{\text{sup}}] \quad (79)$$

$$\pi_\varepsilon \left(\frac{M}{N} - x_\varepsilon^s \right) = 0 \quad \forall \varepsilon \in [0, \varepsilon_{\text{sup}}] \quad (80)$$

$$\frac{\omega - 1}{\beta} = z(1 - N) \int_0^{\varepsilon_{\text{sup}}} \lambda_\varepsilon dF(\varepsilon) + \omega \quad (81)$$

There are two additional conditions relative to the first-order conditions in the text. Equation (78) states the marginal cost of production must be equal to the marginal utility of the money received by the seller. Condition (78) is the additional Kuhn-Tucker condition.

The equilibrium offers. At the symmetric equilibrium, which we will focus on, the values for the different variables of the household i are just equal to the values of the same variables for the other households. Consequently, $\omega = \Omega$, $x_\varepsilon^i = X_\varepsilon^i$, and $q_\varepsilon^i = Q_\varepsilon^i$, where $i = b, s$. Equations (70), (71), (73), and (74) can be rewritten as follows:

$$-q_\varepsilon^b + x_\varepsilon^b \omega = (1 - \theta \Delta) [-q_\varepsilon^s + x_\varepsilon^s \omega] \quad (82)$$

$$\varepsilon u(q_\varepsilon^s) - x_\varepsilon^s \omega = (1 - (1 - \theta) \Delta) [\varepsilon u(q_\varepsilon^b) - x_\varepsilon^b \omega] \quad (83)$$

First, we show that for any ε , λ_ε and π_ε are either both positive or both zero. Suppose, on the contrary, that $\lambda_\varepsilon > 0$ and $\pi_\varepsilon = 0$. From (77) and (78), $\varepsilon u'(q_\varepsilon^s) = 1$ and $\varepsilon u'(q_\varepsilon^b) > 1$. Hence, $q_\varepsilon^s > q_\varepsilon^b$. Furthermore, according to (79), $\lambda_\varepsilon > 0$ implies that $x_\varepsilon^b = m/N$. From (82), when Δ approaches 0 we have $q_\varepsilon^s - q_\varepsilon^b = [x_\varepsilon^s - \frac{m}{N}] \omega$. The left-hand side is positive, whereas the right-hand side cannot be positive because $x_\varepsilon^s \leq \frac{m}{N}$ ($\pi_\varepsilon = 0$). A similar contradiction appears when we assume that $\lambda_\varepsilon = 0$ and $\pi_\varepsilon > 0$. Hence, λ_ε and π_ε are either both positive or both zero.

First case: If $\lambda_\varepsilon > 0$ and $\pi_\varepsilon > 0$, the constraints on money holdings are binding and therefore $x_\varepsilon^s = x_\varepsilon^b = \frac{m}{N}$. Equations (82) and (83) yield

$$\begin{aligned} q_\varepsilon^s - q_\varepsilon^b &= -\theta \Delta \left[-q_\varepsilon^s + \frac{m}{N} \omega \right] \\ \varepsilon u(q_\varepsilon^s) - \varepsilon u(q_\varepsilon^b) &= -(1 - \theta) \Delta \left[\varepsilon u(q_\varepsilon^b) - \frac{m}{N} \omega \right] \end{aligned}$$

The first equation implies that $\lim_{\Delta \rightarrow 0} q_\varepsilon^s = \lim_{\Delta \rightarrow 0} q_\varepsilon^b = q_\varepsilon$. Divide the second equation by the first, and take the limit as $\Delta \rightarrow 0$, to get

$$\varepsilon u'(q_\varepsilon) = \frac{(1 - \theta) \left[\varepsilon u(q_\varepsilon) - \frac{m}{N} \omega \right]}{\theta \left[-q_\varepsilon + \frac{m}{N} \omega \right]} \quad (84)$$

Note that the q_ε that solves (84) also maximizes the asymmetric Nash product

$$q_\varepsilon = \arg \max_q \left[\varepsilon u(q) - \frac{m}{N} \omega \right]^\theta \left[-q + \frac{m}{N} \omega \right]^{1-\theta}$$

This confirms our interpretation that θ is the buyer's bargaining power.

Second case: If $\lambda_\varepsilon = 0$ and $\pi_\varepsilon = 0$, equations (77) and (78) imply $q_\varepsilon = q_\varepsilon^*$. Moreover, when $\Delta \rightarrow 0$, (82) and (83) imply that the amount of money exchanged, x_ε , satisfies

$$-q_\varepsilon^* + x_\varepsilon \omega = (1 - \theta) \left[\varepsilon u(q_\varepsilon^*) - q_\varepsilon^* \right] \quad (85)$$

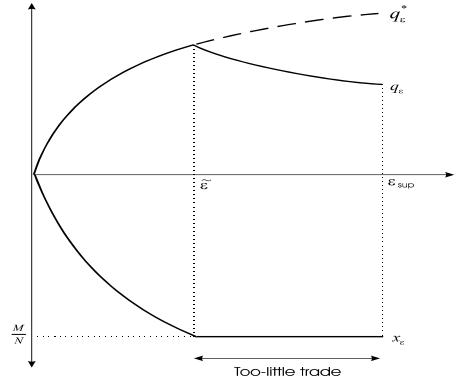


Figure B4.

The inefficiencies (see Figure B4). Let q_ε^c denote the value of q_ε that satisfies (84). One can show that q_ε^c is a decreasing function of ε . Consequently,

$$q_\varepsilon = \min(q_\varepsilon^*, q_\varepsilon^c)$$

Let's define $\tilde{\varepsilon}_D$ as the value of ε such that $q_\varepsilon^* = q_\varepsilon^c$. For all $\varepsilon < \tilde{\varepsilon}_D$ we have $q_\varepsilon = q_\varepsilon^*$ and $x_\varepsilon < \frac{M}{N}$, and for all $\varepsilon > \tilde{\varepsilon}_D$, $q_\varepsilon < q_\varepsilon^*$ and $x_\varepsilon = \frac{M}{N}$. This confirms our conjecture that the only inefficiency left with divisible money is the too-little-trade inefficiency.

Determination of $\tilde{\varepsilon}_D$. When $\varepsilon = \tilde{\varepsilon}_D$, we have $q_{\tilde{\varepsilon}_D} = q_{\tilde{\varepsilon}_D}^*$ and $x_{\tilde{\varepsilon}_D} = \frac{m}{N}$. Thus, the pair $(\tilde{\varepsilon}_D, q_{\tilde{\varepsilon}_D})$ satisfies

$$\tilde{\varepsilon}_D u'(q_{\tilde{\varepsilon}_D}) = 1 \quad (86)$$

$$-q_{\tilde{\varepsilon}_D} + \frac{m}{N} \omega = (1 - \theta) [\tilde{\varepsilon}_D u(q_{\tilde{\varepsilon}_D}) - q_{\tilde{\varepsilon}_D}] \quad (87)$$

In the plane $(q_{\tilde{\varepsilon}_D}, \tilde{\varepsilon}_D)$, (86) gives an upward-sloping curve, whereas (87) gives a downward-sloping curve. Furthermore, the curve given by (87) moves upward as $m\omega$ increases. As a consequence, $\tilde{\varepsilon}_D$ is increasing in the real stock of money, $m\omega$.

The marginal value of money ω . Using the fact that for all $\varepsilon \leq \tilde{\varepsilon}_D$ one has $\lambda_\varepsilon = 0$, the envelope condition (81) can be rewritten as follows:

$$\frac{\omega_{-1}}{\beta} = z(1 - N) \int_{\tilde{\varepsilon}_D}^{\varepsilon_{\sup}} \lambda_\varepsilon dF(\varepsilon) + \omega$$

Divide this equation by ω , substitute for $\lambda_\varepsilon/\omega$ its expression given in (77), and rearrange to get

$$\frac{\frac{(\omega_{-1}/\omega)}{\beta} - 1}{z(1 - N)} = \int_{\tilde{\varepsilon}_D}^{\varepsilon_{\sup}} [\varepsilon u'(q_\varepsilon^c) - 1] dF(\varepsilon) \quad (88)$$

In the steady state, the real value of money holdings, ωm , is stationary. Therefore, $\frac{m}{m_{-1}} = \frac{\omega_{-1}}{\omega} = \gamma$. Consequently, from (88), ωm is determined by

$$\int_{\tilde{\varepsilon}_D}^{\varepsilon_{\sup}} [\varepsilon u'(q_\varepsilon^c(\omega m)) - 1] dF(\varepsilon) = \frac{\frac{\gamma}{\beta} - 1}{z(1 - N)} \quad (89)$$

Let $(m\omega)^*$ denote the value of $m\omega$ such that $\tilde{\varepsilon}_D = \varepsilon_{\sup}$. The derivative of the LHS of (89) with respect to $m\omega$ is

$$\frac{\partial LHS}{\partial m\omega} = \int_{\tilde{\varepsilon}_D}^{\varepsilon_{\sup}} \varepsilon u''(q_\varepsilon^c) \frac{\partial q_\varepsilon^c}{\partial m\omega} dF(\varepsilon) < 0 \quad \text{for all } m\omega < (m\omega)^*$$

because from (84) $\frac{\partial q_\varepsilon^c}{\partial m\omega} > 0$. Furthermore, $LHS(0) = +\infty$ and $LHS((m\omega)^*) = 0$. As a consequence, there is a unique $m\omega \in (0, (m\omega)^*)$ that satisfies (89).

Efficiency. According to (86), (87), and (89), for all $\gamma < \beta$ we have $m\omega < (m\omega)^*$ and $\tilde{\varepsilon}_D < \varepsilon_{\sup}$. This implies that the too-little-trade inefficiency is always present if $\gamma > \beta$. If $\gamma \rightarrow \beta$, then $m\omega \rightarrow \beta$ and $\tilde{\varepsilon}_D \rightarrow \varepsilon_{\sup}$. Efficiency in the decentralized economy requires $\tilde{\varepsilon}_D = \varepsilon_{\sup}$, which is achieved by $\gamma = \beta$. The Friedman rule holds.

Literature

- Berentsen, A., 2000. Money inventories in search equilibrium, *Journal of Money, Credit, and Banking* 32, 168–178.
- Berentsen, A., Molico, M. and R. Wright, 2000. Indivisibilities, lotteries, and monetary exchange. *Journal of Economic Theory*, forthcoming.
- Berentsen, A. and G. Rocheteau, 2000. Money and terms of trade. *International Economic Review*, forthcoming.
- Berentsen, A. and G. Rocheteau, 2001. A note on the Friedman rule in search models of money. Mimeo.
- Berentsen, A., Rocheteau, G. and S. Shi, 2001. Hosios meets Friedman: On efficiency in search models of money. Mimeo.
- Boldrin, M., Kiyotaki, N. and R. Wright, 1993. A dynamic equilibrium model of search, production, and exchange. *Journal of Economic Dynamics and Control* 17, 723–758.
- Green, E.J. and R. Zhou, 1998. A rudimentary random-matching model with divisible money and prices. *Journal of Economic Theory* 81, 252–271.
- Jafarey, J. and A. Masters, 2000. Output, prices and the velocity of money in search equilibrium. Mimeo.
- Jevons, W.S., 1875. *Money and the mechanism of exchange* (Appleton, London). First chapter reprinted in *General equilibrium models of monetary economics: Studies in the static foundations of monetary theory*. Ross Starr (Ed.). Academic Press, Inc. (1989).
- Kiyotaki, N. and R. Wright, 1991. A contribution to the pure theory of money. *Journal of Economic Theory* 53, 215–235.
- Kiyotaki, N. and R. Wright, 1993. A search-theoretic approach to monetary economics. *American Economic Review* 83, 63–77.
- Lagos, R. and R. Wright, 2001. A unified framework for monetary theory and policy analysis. Mimeo.
- Li, V., 1995. The optimal taxation of fiat money in search equilibrium. *International Economic Review* 36, No. 4, 927–942.
- Lucas, R.E., 1990. Liquidity and interest rates. *Journal of Economic Theory* 50, 237–264.
- Molico, M., 1996. The distribution of money and prices in search equilibrium. Mimeo, University of Pennsylvania.
- Rauch, B., 2000. A divisible search model of fiat money: A comment. *Econometrica* 68, 149–156.
- Muthoo, A., 1999. *Bargaining theory with applications* (Cambridge University Press, Cambridge).

- Osborne and A. Rubinstein, 1990. *Bargaining and markets* (Academic Press, San Diego).
- Rocheteau, G., 2000. La quantité optimale de monnaie dans un modèle avec appariements aléatoires, *Les Annales d'Economie et Statistique* 58, 101–142.
- Shi, S., 1995. Money and prices: A model of search and bargaining. *Journal of Economic Theory* 67, 467–496.
- Shi, S., 1997. A divisible search model of fiat money. *Econometrica* 65, 75–102.
- Shi, S., 1999. Search, inflation and capital accumulation. *Journal of Monetary Economics* 44, 81–103.
- Shi, S., 2001. Liquidity, bargaining, and multiple equilibria in a search monetary model. *Annals of Economics and Finance* 2, 191–217.
- Taber, A. and N. Wallace, 1999. A matching model with bounded holdings of indivisible money. *International Economic Review* 40, 961–984.
- Trejos, A. and R. Wright (1995). Search, bargaining, money, and prices. *Journal of Political Economy* 103, 118–141.
- Wallace, N., 1997. Absence-of-double-coincidence models of money: A progress report, *Federal Reserve Bank of Minneapolis Quarterly Review* 21, 2–20.
- Wright, R., 1999. Theory of monetary exchange II : Search and bargaining models of exchange. Lecture notes.
- Zhou, R., 1999. Individual and aggregate real balances in a random-matching model. *International Economic Review* 40, 1009–1038.